ELEMENTARY
[ME]CHANICS OF SOLIDS

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SPECIMEN

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BY

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PREFACE.

This book has been written after many years' experience in teaching Theoretical Mechanics to students of a great variety of ages and attainments.

I attach great importance to the value of carefully selected and carefully explained examples, and throughout the book numerous examples will be found worked out and accompanied by notes on the processes employed in their solution. In addition to these, there are nearly five hundred questions for exercise. Many of these questions are taken from examination papers, the source being always clearly stated.

In the teaching of any branch of science the value of experimental illustrations has now come to be fully recognized; and I have described upwards of forty experiments which may all be performed by teacher or student with the help of very inexpensive apparatus. At the same time more elaborate apparatus may be used, when it is available, to illustrate much of the subject matter.

The book will be found to contain all the subjects in the syllabus of the Elementary Stage of Theoretical Mechanics of Solids of the Board of Education, South Kensington; while, to increase its usefulness and to adapt it to the requirements of students for other examinations, the theoretical proofs of many propositions have been added.

It may be read without any mathematical attainments beyond an ability to solve easy algebraical equations, except that in a few instances easy quadratics and the properties of similar triangles have been employed.

My thanks are due to Prof. R. A. Gregory and Mr. A. T. Simmons, B.Sc., who have helped me by many valuable suggestions during the preparation of the book.

W. T. A. EMTAGE.

London, *July*, 1900.

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CHAPTER I.

FORCE. PARALLELOGRAM AND TRIANGLE OF FORCES.

Force.—Suppose a piece of wood is placed on a smooth horizontal table, or, better still, to float on water, so that it will yield to the application of the slightest push or pull in any direction. Now let two strings be attached to it, and let these both be pulled out horizontally. As a rule the wood will yield to the combined effect of the pulls in the strings, or the two forces applied to it, and will begin to move. It is easy to imagine in a general way what will be the effect produced.

(1) If the two pulls are inclined to one another, the wood will begin to move off along a line lying in the angle between them.

(2) If they are opposite to one another, the wood will move in the direction of the greater pull.

(3) The wood may begin to turn instead of moving away bodily, or even perhaps as well as moving away bodily.

Suppose the two pulls to be equal to one another, that is, so that the tensions in the two strings are equal, and suppose that they act in opposite directions along parallel straight lines, not along the same straight line. The combined effect of these two pulls will be to turn the wood round without moving it away bodily.

Figures 1, 2, 3 represent these three cases. The arrow-heads **P** and **Q** denote the pulls of the strings; and *M* denotes the motion of the wood.

Now in certain circumstances the wood will not move at all. Let us consider what these are. The two pulls must be along the same straight line; that is, the first string, the line joining the two points of attachment, and the second string must be all

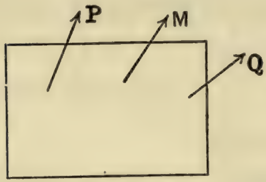

Fig. 1.

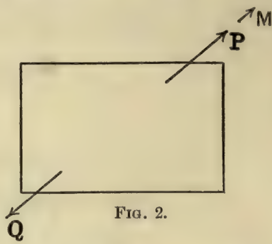

Fig. 2.

in one straight line : the two pulls must be equal to one another: they must act in opposite senses. It is only under these conditions that the wood will remain at rest. And, further, whenever these conditions are fulfilled, we may be sure that the combined effect of the two forces acting on the wood will be nothing.

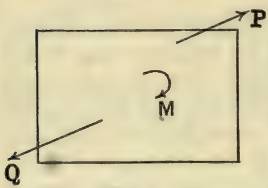

Fig. 3.

In what we have just considered the pulls in the strings are examples of *mechanical forces.* Such forces are produced in numerous ways ; and in general we may say :

A force is that which moves, or tends to move, a body, or alters, or tends to alter, its state of motion.

Equilibrium.—If a body is acted on by a set of forces in such a manner that it does not move it is said to be *in equilibrium.* Sometimes the **forces** are spoken of as being **in equilibrium,** or are said to form a system in equilibrium with each other, this meaning that their combined effect on any body on which they may be acting is nothing.

Conditions for Equilibrium.—We have just met with an example of forces in equilibrium, the case in which the forces are two in number. Let us now state, in general terms, the conditions which must *necessarily* hold when two forces are in equilibrium, and which are *sufficient* to ensure that the forces shall be in equilibrium. The conditions may thus be stated to be necessary and sufficient.

In saying that they are necessary, we say that if we know that the forces are in equilibrium, the conditions must hold, or must necessarily hold; and in saying that they are sufficient, we say that if the conditions are known to hold, the forces must be in equilibrium, or the conditions suffice to ensure equilibrium. It should be remembered, then, that when conditions are said to be necessary and sufficient, two distinct statements are made: in each of them we know something, and something else follows as a result; and what we know in one case is what follows in the other, and *vice versa*. The two statements, or propositions, are thus converses of each other. We may now say that

The necessary and sufficient conditions between two forces in equilibrium are that they should be equal, and should act in opposite directions along the same straight line.

Thus, in the case of the wood pulled by two strings, when we say what conditions are necessary, we mean that if the wood acted on by the two pulls does not move, the pulls *must* be equal and act oppositely along the same straight line; and when we say what conditions are sufficient, we mean that if the pulls are equal and act oppositely along the same straight line, the wood *will not* move.

It is important to understand this about necessary and sufficient conditions, because it frequently happens that, in a case of this sort, the two sets of conditions are the same, and we thus have a compact way of stating what they are.

Transmissibility of Force.—We have seen that the force **P** balances another force **Q**, if it is equal to **Q** and acts along the same straight line in the opposite sense. And this is entirely independent of the point of application of the force **P**, so long as it is some point of the body on which the forces act, and is in the straight line in which **Q** acts.

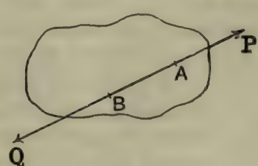

FIG. 4.

If, for instance, **P** is a pull due to a string, the end of the string may be attached to any point in the straight line of **Q**'s action, and then if **P** pulls in exactly the opposite direction to **Q**, and the forces are equal, no motion will be produced in the body.

It follows that we may, for statical purposes, that is, so far as tendency to move a body is concerned, suppose a force to act at any point we please in its line of action.

This is called the principle of the *Transmissibility of Force.*

Tension of Strings.—We have seen that a force may be caused to act on a body by attaching a string to it and pulling the string, as, for instance, with the hand. The pull exerted by the hand is transmitted along the string, and is applied to the body. The string is said to be in a state of tension. The pull all along its whole length, or the force which any piece of it exerts on the next piece, is the same, being equal to the force applied by the hand. This pull, which is exerted throughout the length of the string, is called the **tension** of the string; and we may say that the force acting on the body is the tension of the string.

A pull exerted by the hand in this way would not be a definite or constant force.

A steady force of a definite magnitude may be obtained in various ways.

If a body is tied to the end of a string and hangs steadily from it, it produces by its weight a constant pull along the string, of a definite magnitude.

If the string passes round a smooth pulley, that is, one which turns quite readily on its axle, the pull throughout the string will still be the same as in the vertical portion of it which is immediately above the body.

Definite Forces.—An elastic string, such as an indiarubber band, may be used to obtain a steady force. The pull necessary to stretch out such a string by a given amount depends on the amount of stretching. Thus, if one end of such a string be attached to a body, and the string be pulled out by a force applied to the other end till a given amount of stretching is produced, a definite force will act on the body depending on the amount of stretching produced in the string.

If we suspend the string by one end, and hang at the other end various weights, 1, 2, 3, etc., ounces, it will be found that the amounts by which these stretch the string are approximately proportional, and 1, 2, 3, etc. Thus, the pull in the string not only depends on, but may be taken as proportional to, the amount by which it is stretched.

A coiled spiral spring behaves in the same way as an elastic string. The pull which stretches it is proportional to the stretching produced. Such a spring may thus be used to indicate forces by observing the stretching produced. This is done in the case of the spring-balance.

EXPERIMENT 1.—Take a band of rubber 5 or 6 inches long, and tie strings to its two ends. Attach one end to a fixed point so that the band hangs vertical. Measure the length between the two points of the band to which the strings are attached. Now hang on it various weights, such as 10, 20, 30, 40, 50 grams. Measure the corresponding stretched length in each case, and so determine the stretching which each weight produces. These should be approximately proportional to the stretching weights.

The weights used will, of course, depend on the strength of the band; the heaviest should not be great enough to injure or permanently elongate it.

If the shape of the weights will not allow them to be readily attached to the string, a pan must be used into which to put them; and the pan may be loaded with small pieces of metal or other material to bring it up to the weight of the smallest weight. Thus, if we are using ounce weights, the pan may be loaded to make it weigh one ounce.

EXPERIMENT 2.—The same experiment may be performed with a spiral spring; and it will again be found that the elongations produced are approximately proportional to the weights used. If the coils of the spring lie in contact with each other to start with, some force may be required to make them begin to separate. Then the elongations are proportional to the *additional* weights used.

EXPERIMENT 3.—Observe the stretching produced in a rubber band or a spiral spring by a certain weight. Fix a pulley that will run very smoothly. Pass a string over it: hang the weight on one side and support it by means of the band or spring attached to the other end of the string passing over the pulley. Notice the elongation produced. This should be about the same as in the first case when the pulley was not used.

Draw the weight down a little, thus slightly further stretching the band, and allow it to go back slowly to its position of rest: and again raise it a little and allow it to come down to its position of rest. It will probably be found that these two positions are not quite the same, giving elongations of the string that differ slightly from each other and from that attained at first. This is due to a little friction in the pulley which cannot be quite got rid of.

NOTE.—For the pulleys used in this and other experiments the light aluminium pulleys now supplied by many makers of scientific apparatus will be found very suitable. They can be obtained, for instance, from Messrs. Griffin & Sons, Sardinia Street, W.C.

Measurement of Forces.—In questions concerning the equilibrium of forces, we have frequently merely to consider the ratios of forces to each other ; but it is also convenient to have some method of measuring forces, that is, some unit of measurement, in terms of which we may specify them by saying how many times any force contains the unit. The unit most frequently employed in such questions is the *weight of a pound.* We thus speak of a force of two, three, etc., pounds' weight. Notice carefully that the weight of a body means the force with which the earth attracts it to itself in a vertically downward direction : the weight of a pound is the force of attraction which acts on a definite quantity, a pound, of matter. And it is accurate to speak of *a force of three pounds' weight*, or, as it is sometimes written, *a force of three lbs.' weight.* Occasionally, however, such an expression as *a force of three pounds* is met with. This is used for the sake of brevity ; but cannot be altogether justified. It is customary in ordinary language, and even in mechanics, to speak of a body of a definite weight as *a weight.* Thus *a weight of* 10 *lbs.* may mean a body, a definite quantity of matter, and not a force at all. No confusion will arise, as a rule, since the context indicates what is meant.

The gram is a mass used in the French or Metric System of measures. It is about $\frac{1}{454}$th of a pound. A force is often expressed in grams' weight.

Graphic Representation of Forces.—It is found to be very convenient to have a means of representing forces in diagrams. They are represented by straight lines.

In the first place a straight line may be drawn to represent the actual line along which the force acts. If, for instance, the force is a pull in a string, the line may be taken to represent, or to be a picture of, the string.

It is, however, often sufficient, and even more convenient, to take a straight line to represent the direction, but not the actual line of action of the force, so that it is then drawn parallel to the line along which the force acts.

In this latter case the line would represent the force in **direction** : and in the former case in *line of action* or **position** as well as in direction.

In either case it is further necessary to indicate the **sense**, or the one of the two ways along the straight line in which the force in question acts. This is frequently done, as we have done it already, in Figs. 1, 2, 3, by means of arrow-heads.

Lastly, the line may be taken to represent the force in magnitude, according to some convenient scale. Thus, we may agree to represent each pound's weight by one inch ; and the line would be drawn as many inches in length as there are pounds' weight in the force. The scale on which to represent the forces would, of course, be chosen according to the magnitudes of the forces in question, and the size of the diagram that it is desired to obtain.

We thus see how a force may be represented, graphically, by means of a straight line in (1) *direction*, (2) *sense*, (3) *line of action*, (4) *magnitude*.

These four points or particulars, which can all be represented on a diagram, make up the complete **specification of a force**.

Point of Application is immaterial.—It should be noticed that nothing is here said about the actual point to which the force is applied, because this is immaterial. If we know the straight line along which a force acts, it would produce exactly the same effect in moving, or tending to move, the body on which it acts, no matter to what particular point in this straight line it is applied. Of course, if the body moves, the line of action may shift into a position depending on the point of application. But, as long as the line of action is given, the point of application is immaterial.

If we say that a force is represented by a straight line AB, the manner of naming the straight line indicates the required sense, which need not then be more particularly specified ; the sense from A to B is indicated. In this way of looking at the matter, we may suppose that the two senses along a straight line are two different, opposite, directions ; and so we may leave out the idea of sense altogether, the complete direction of the force being specified by the way of drawing the line, and the way of naming it. Also, as has already been said, for many purposes it is not necessary to indicate the actual line along which a force acts, but merely its direction. In these cases then, we indicate a force sufficiently if we represent it by a straight line *in direction and in magnitude.*

In the figure let **F** denote a force of 5 pounds' weight. Let us choose a scale of $\frac{1}{4}$-in. to a pound's weight. Draw a straight line AB parallel to the line of action of **F**, and make AB $1\frac{1}{4}$-ins. long. Then the force **F** is represented in direction and magnitude by AB.

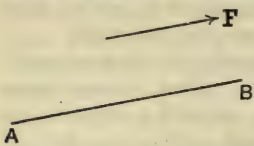

FIG. 5.—Line representing a force in direction and magnitude.

Note that **F** is not represented in position or line of action by AB. Nor is **F** represented in direction by BA, but by AB.

Resultant.—If a given set of forces can be replaced by a single one, this is called their *resultant*. The resultant is thus a force which produces exactly the same effect as the given forces. It is clear that it must be specified, not only in magnitude, but in all particulars. Thus, if given forces have for resultant a force of a certain magnitude acting in a certain manner, a force of the same magnitude acting along some other line would not be their resultant.

Equilibrant.—If a set of forces can be held in equilibrium by a single one, this is called their *equilibrant*. The force which will hold the given forces in equilibrium will clearly also hold their resultant in equilibrium, since the resultant produces just the same effect as the given forces. Since the resultant and the equilibrant just balance each other, it follows that they must be equal forces, and act in opposite directions along the same straight line. That is, they differ in nothing but *sense*.

We now come to a very important proposition, by means of which we can determine the resultant of any two forces which are inclined to each other, that is, whose lines of action are not parallel but intersect. This proposition is called the *Parallelogram of Forces*. It forms the basis of the science of Statics. It may be stated as follows :

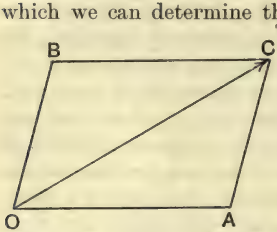

FIG. 6.—Parallelogram of forces.

Parallelogram of Forces.—*If two forces are represented in direction and magnitude by the two straight lines OA, OB drawn from the point O, then their resultant will be represented in direc-*

tion and magnitude by the diagonal OC of the parallelogram OACB described on OA, OB as two adjacent sides.

Experimental Verification.—*First Method.*—This proposition may be verified experimentally in the following way. Three fine strings are knotted together at the point *C*, and to their other ends known weights, **P**, **Q**, **R** are attached. The strings *P*, *Q* are passed over smooth pulleys *A*, *B*, so that the tension throughout either of these strings is then uniform, and equal to the weight of the body at the end of it. Thus at *C* three forces are acting along the strings equal to the three weights. Now, by placing a black-board or a sheet of paper close behind the strings (a very convenient plan being to attach the pulleys to such a board), distances *CE*, *CD* may be marked off on it just behind the strings to represent, according to a chosen scale, the weights of **P** and **Q**. Complete the parallelogram *CEFD*, and draw the diagonal *CF*.

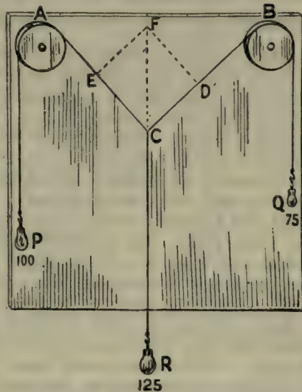

FIG. 7.—Experimental verification of parallelogram of forces.

P and **Q** along *CE* and *CD* are held in equilibrium by **R** vertically downwards; so that we know that their resultant is **R** vertically upwards. We have therefore to see whether the construction of the parallelogram of forces gives this result. This construction gives *CF* as representing the resultant. *CF* ought therefore

(1) to be vertical, and

(2) by its length to represent **R** on the same scale as *CE* and *CD* represent **P** and **Q**.

If the construction is carefully made, *CF* will be found to satisfy these conditions.

Second Method.—The experiment may also be carried out by using spring balances instead of pulleys and known weights. Two such balances are used in the arms *CE*, *CD* being fastened to pegs at A and B. A known weight may be used at **R**, or a third balance may be attached to the string C**R**, and pulled out to give any desired indication. The string C**R** will then not necessarily be vertical. But CF must be in the production of the line of this string, and represent by its length the indication of the balance in C**R** on the same scale as CE and CD represent the pulls shown by the other two balances.

EXPERIMENT 4.—Attach two smooth pulleys to the top corners of a black-board, and fix the black-board vertical. Tie three strings together by an end of each, and pass two of them over the pulleys, letting the third hang down. To the other ends of the strings attach weights. These must be so chosen that the knot of the strings will come to rest at some point in front of the board. Any combination of weights whatever will not do, as in some cases the knot would run over a pulley. Weights 20 and 30 grams at the sides and 40 in the middle may be used. Note the point at which the knot rests. On account of the friction of the pulleys it will be found that there is a little range, a small area, at any point of which the knot may be made to rest. The best position for it is about the mean position of this range, or centre of the area.

From this point draw straight lines just behind the three strings. Mark off along the lines lengths to represent the forces acting in the strings. With a fair sized black-board, and the weights mentioned, 3 inches may be taken for each 10 grams' weight. So that the lengths would be 6, 9, 12 inches.

Construct a parallelogram on the lines 6 and 9 inches, and draw its diagonal from the position of the knot. This should represent the resultant of the weights of the 20 and 30 grams. It should therefore represent a force equal and opposite to the 40 grams' weight which balances the other two.

Thus, the diagonal should be in a straight line with the 12 inch line, and should be 12 inches long.

EXPERIMENT 5.—Take three rubber bands and tie their ends to six strings. Fasten three of the strings together in a knot, and pull out the other strings so as to stretch out the bands over a piece of drawing paper on a drawing-board, fastening the other ends with the bands in the stretched positions.

The tensions in the bands are three forces in equilibrium acting along the strings which meet at the knot. Draw three straight lines out from the position of the knot to mark the directions of the strings. Carefully measure the stretched lengths of the bands. Remove the bands and find what weights they must carry to stretch them to the same lengths. These denote the forces that acted along the strings. Measure off along the lines of the strings portions proportional to these forces. Thus, if the largest weight is 25 grams, on an ordinary drawing-board a scale of 1 inch to 5 grams' weight will probably be found suitable.

Construct the parallelogram on two of the lines, and find its diagonal through the knot. This should be equal and opposite to the third line.

Measurement of Angles.—In assigning the relative positions of two forces, or of two straight lines, the angle between them is usually specified in degrees and fractions of a degree.

Let *ACBD* be a circle with centre *O*.

Imagine the circumference to be divided up into 360 equal parts. The straight lines joining *O* to the points of division form 360 equal little angles. Each of these angles is a **degree.**

FIG. 8.

It is clear that 360 such degrees fill up the whole space round *O*; and if *AOB*, *COD* are two diameters at right angles to each other, each of the four right angles at *O* contains 90 degrees.

We may say that a degree is $\frac{1}{90}$th part of a right angle. Thus, if the right angle were divided into 90 equal little angles, each would be a degree.

The size of the degree does not depend at all on the size of the circle used to obtain it. The inclination of the two straight lines containing a degree would be the same whatever the size of the circle.

An angle of one degree is written 1°.

Thus, one right angle = 90°.

Protractors.—The instrument used for measuring and for laying out angles is called a protractor. In the figure two forms of protractor are shown. In the outer semicircular one the angles from 0° to 180° are marked off with the point * as centre.

To measure an angle, the centre of the protractor is placed at the vertex of the angle, that is, the point where its lines meet, and the line to the mark corresponding to 0° along one of the arms of the angle. The graduation coming over the other arm gives the required size.

Similarly, to construct an angle of given size, the centre is placed at the required vertex, and having drawn one arm, by the help of the graduations, the position for the other arm is found.

The inner rectangular instrument has a marked point for centre, and the graduations are placed along its edges. Its use is quite similar to that of the other one. It is not calculated to give such accurate results.

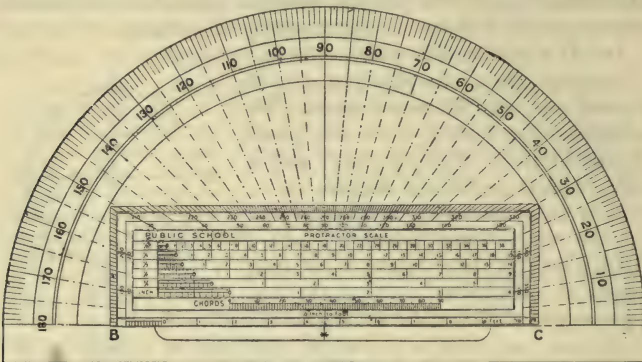

FIG. 9.—Protractors.

Scales.—For the examples that will be given here, a scale marked in inches and 10ths is very useful. One marked in 8ths may be used, and the results reduced when necessary.

Lengths are sometimes expressed in metres and centimetres. The metre is the standard of length in the metric system, and is about 39 inches. The centimetre is $\frac{1}{100}$ of a metre, and is therefore about $\frac{2}{5}$ of an inch.

How the Method of finding Resultants is applied.—As an example of finding a resultant by means of the parallelogram of forces, let us consider this question :

Find the position and magnitude of the resultant of two forces of 5 and 6 lbs. weight, inclined at an angle of 65°.

This question may be solved by accurate drawing and measuring by means of a rule and protractor for the lengths and angles. We must first decide on a scale of lengths for representing the forces, that is, decide what length is to be taken to represent a unit of force, or in this case a pound's weight. It must be remembered that in solving a question in this way, that is, graphically, in order to obtain an accurate result, we must use a pretty large scale, and draw pretty large figures, because a length of about 10 or 12 inches can be laid off and measured with the rule to greater proportionate, or per centage, accuracy than one of an inch or two. In the present case we may use an inch length to denote a pound's weight. We should then draw two straight lines OA, OB, 6 and 5 inches long, making an angle AOB, as measured by the protractor, equal to 65°. Completing the parallelogram, and drawing the diagonal OC, it will be easy to find that, correct to $\frac{1}{10}$th of an inch for length, and 1° for angular measure, OC is 9·3 inches long, and makes an angle of 29° with OA.

The required result will then be:

The resultant makes angles 29° and 36° with the given forces, and is a force of 9·3 lbs.' weight.

The figure drawn of the size here suggested would more than fill a page of this book.

The following examples are intended to be solved in the same way, that is, by means of careful drawing and measuring. The results are given approximately. They profess to be correct as far as they are given, but not to be quite exact. Thus, in the result 9·3 lbs.' weight, given above, it is understood that this is correct to the first place of decimals; and the correct result is nearer to 9·3 than to 9·2 or 9·4.

Exercises I. a.

Find the magnitudes and positions of the resultants in the following cases:

1. Forces of 97 and 90 units inclined at 134°.
2. Forces of 11 and 6 lbs.' weight, making an angle of 66°.
3. 10 and 17 units at right angles.

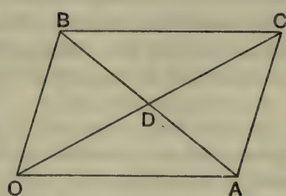

Fig. 10.—Method of finding resultant of two forces.

Another Method of finding a Resultant.—The following is another method by which the resultant may be more readily obtained.

Since the diagonals of any parallelogram bisect each other, it follows that if we draw the line AB and bisect it at D, the required diagonal OC is equal to

$2OD$, and its direction is known since it is that of OD. It is therefore not necessary to draw the complete parallelogram: we need only complete the triangle OAB, in which OA, OB represent the given forces, and draw the *median* OD to the point of bisection of AB. OD then gives the direction of the required resultant, and this resultant is represented by $2OD$.

The Triangle of Forces.—In the parallelogram of forces OA, OB represent two forces, and OC their resultant. Therefore the forces OA, OB would be neutralized by a force represented by CO. Or the three forces OA, OB, CO acting at the point O are in equilibrium.

Now these three forces are completely represented by the lines OA, OB, CO; but since AC is equal and parallel to OB, AC represents the second force in direction and magnitude, although not in position.

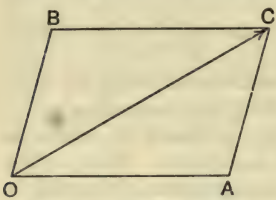

FIG. 11.—Triangle of forces

Hence the two given forces acting at O are represented in direction and magnitude by the two sides OA, AC of the triangle OAC; and the force which is represented by CO neutralizes them.

The three forces are represented by the sides of the triangle OAC, named the same way round, that is, *taken in order*.

We have the conclusion:

If three forces acting at a point can be represented in direction and magnitude by the sides of a triangle taken in order, the forces are in equilibrium.

Again, if we know that three forces acting at a point are in equilibrium, we may take OA, OB to represent two of them. Then, since the resultant of these is OC, the third force must be represented by CO. Hence the three forces can be represented in direction and magnitude by OA, AC, CO.

And we conclude that:

If three forces acting at a point are in equilibrium, they can be represented in direction and magnitude by the three sides of a triangle taken in order.

These two results are two converse propositions, the first of which is known as *The Triangle of Forces.*

These results may be stated in a more compact form. But first notice that the triangle of forces gives us nothing about the relative positions or the lines of action of the forces. For equilibrium, it is clear, by the parallelogram of forces, that any force to be the equilibrant of the other two must pass through their point of intersection. Hence, if we know that three forces pass through a point, the first conclusion given above tells us that for the forces to be capable of being represented by the sides of a triangle taken in order is a sufficient condition for equilibrium; and the second conclusion tells us that it is a necessary condition.

Results.—We may then state the results:

The necessary and sufficient conditions for three forces in equilibrium are

(1) *That they should pass through a point.*

(2) *That they should be capable of being represented by the three sides of a triangle taken in order.*

There is one exception to the first condition that the three forces should all pass through one point; that is, when the three forces are all parallel to each other. What the conditions are in this case we shall see later on. But if two of the forces are inclined, so that their lines meet at a point, as considered above, then it is always necessary for equilibrium that the third force should pass through the same point.

Practical Application of Results.—These results are of great use in practice. The representation of forces by means of the sides of triangles is more convenient than the use of the Parallelogram of Forces.

Any question of finding the resultant of two given forces can be solved more conveniently by drawing a triangle. The only thing that the triangle does not give is the position of the resultant; but this is known at once when the direction has been found, since the resultant must pass through the point of intersection of the two given forces.

The order of drawing the sides of the triangle is of great importance, and great care must be exercised in regard to it. Referring to the figure of the parallelogram of forces, we see that the lines OA, OB are drawn out of the same point O; but in the triangle the lines OA, AC, which represent the two forces, are not drawn out of the same point; *one begins where the other stops*, and the angle between them is not the angle between the given forces, but the angle supplementary to it, that is, the adjacent angle, which with the given one makes two right angles. The line representing the equilibrant of these two is CO, which closes up the triangle and brings us back to the starting point. The resultant of the two given forces is OC, which carries us from O to C, just as the paths OA, AC do, only along a straight course. The line of the resultant is the line drawn from the starting point of one of the lines representing the forces to the stopping point of the other; or it has the same starting point as one, and the same stopping point as the other.

For the same two given forces another triangle may be drawn by drawing OB to represent one force, and then BC to represent the other, so that we get the other half of the parallelogram. We obtain, of course, the same results; CO for equilibrant, and OC for resultant.

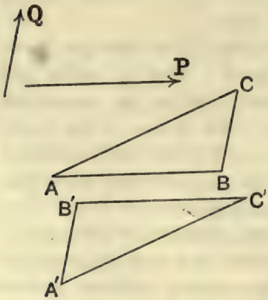

FIG. 12.—Two triangles for given pair of forces.

If **P** and **Q** are two given forces, the figures show how we may construct a triangle to obtain their resultant, namely, either by drawing AB first for **P**, and then BC for **Q**, or by drawing $A'B'$ first for **Q**, and then $B'C'$ for **P**. The resultants AC, $A'C'$, obtained in the two cases, represent the same force in magnitude and direction.

The following examples should be solved by accurate measuring and drawing of the triangle representing the given forces and their resultant.

Exercises I. b.

Find the resultants in the following cases.

1. Forces of 43 and 37 tons weight inclined at 126°.
2. 40 and 69·3, making an angle of 150°.

Certain Simple Results.—In all cases the resultant of given forces can be found by calculation or construction. There are, however, certain simple cases of right-angled triangles that occur so frequently in mechanical questions that the relations between their parts should be carefully noted.

If ABC is a triangle having C a right angle and each of the other angles 45°, then the sides are to one another as 1, 1, $\sqrt{2}$.

Thus, $$\frac{BC}{1}=\frac{CA}{1}=\frac{AB}{\sqrt{2}}.$$

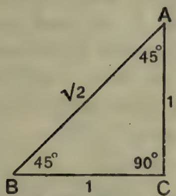

FIG. 13.

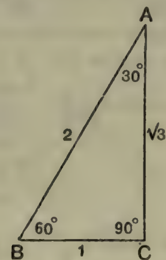

FIG. 14.

If ABC is a triangle having the angles 30°, 60°, and 90°, then the ratios of the sides are given by

$$\frac{CB}{1}=\frac{BA}{2}=\frac{AC}{\sqrt{3}}.$$

It will be convenient to notice for the sake of numerical examples that, approximately, $\sqrt{2}=1{\cdot}414$, $\sqrt{3}=1{\cdot}732$.

When by any means we can calculate the length of the diagonal of the parallelogram in terms of the two sides, or the third side of the triangle in terms of the other two, we can determine the magnitude of the resultant of two given forces.

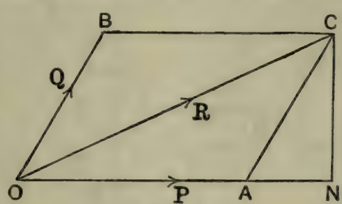

FIG. 15.

Suppose for example, that OA, OB represent two forces **P** and **Q**, making an angle of 60°, and OC their resultant. Draw CN perpendicular to OA produced.

Then, by Euclid I. 47, $OC^2 = ON^2 + NC^2$.

But $$AN = \tfrac{1}{2}AC, \text{ and } NC = \frac{\sqrt{3}}{2} . AC.$$

$$\therefore\ OC^2 = (OA + \tfrac{1}{2}AC)^2 + \left(\frac{\sqrt{3}}{2}AC\right)^2$$
$$= OA^2 + AC^2 + OA . AC.$$

Now since OA, AC, OC contain as many units of length respectively as **P**, **Q**, **R** contain units of force, we may write for this equation $\mathbf{R}^2 = \mathbf{P}^2 + \mathbf{Q}^2 + \mathbf{PQ}$.

In a similar way the magnitudes of the resultants of forces containing angles 30°, 45°, 120°, 135°, 150° may be found by the help of Euclid I. 47, and the simple right-angled triangles mentioned above.

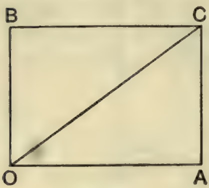

FIG. 16.

In the case in which the given forces **P** and **Q** are at right angles, since

$$OC^2 = OA^2 + AC^2,$$
$$\mathbf{R}^2 = \mathbf{P}^2 + \mathbf{Q}^2.$$

The following examples will illustrate the conditions for forces in equilibrium.

EXAMPLE.—A body weighing 1 cwt. is suspended at the end of a rope. It is tied to another rope which is pulled out horizontally till the first becomes inclined at 30° to the vertical. Find the tensions in the two ropes.

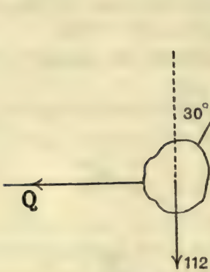

FIG. 17.

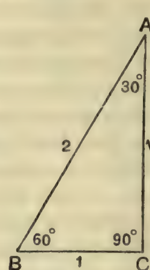

FIG. 18.

The figure represents the arrangement. The body is acted upon by three forces, its own weight, which is 112 lbs.' wt., and acts vertically downwards, and the pulls or tensions in the two ropes. Call these **P** and **Q** lbs.' wt. Now, if we draw a triangle *ABC* with its sides parallel to the lines of action of these three forces, the sides will also be proportional to the three forces. Also we know the ratios of the sides of the triangle *ABC*; so that we know the ratios of the forces; and since we know one of these we can calculate the other two, knowing the relations which they bear to the known one.

Thus, since the forces **P** lbs.' wt. and 112 lbs.' wt. are represented by the lines *BA* and *AC*, the ratio of the forces is the same as that of the lines.

Or $$\frac{\mathbf{P}}{112}=\frac{2}{\sqrt{3}}.$$

From which $$\mathbf{P}=\frac{224}{\sqrt{3}}=129{\cdot}3.$$

$$\frac{\mathbf{Q}}{112}=\frac{1}{\sqrt{3}};\quad \mathbf{Q}=64{\cdot}7.$$

The answers are given correct to the first decimal place, or to $\frac{1}{10}$th of a lb's. weight.

It will generally be found convenient to use one figure for the working in a case of this sort instead of two; that is, instead of drawing the triangle of forces as a separate figure, it is drawn as an addition to the figure which represents, or is a picture of, the arrangement in the question.

We shall now show how this may be done for this question, and, at the same time, give a formal solution of the question such as would be required in answer to it.

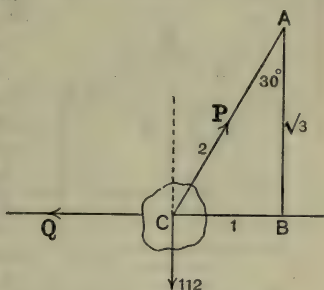

FIG. 19.

Let **P** and **Q** lbs.' wt. be the tensions in the two ropes.

The body is acted upon by these tensions and its own weight acting vertically downwards.

Let C be the point in which these three forces meet.

From A, a point on the first rope, draw the vertical AB to meet the line of the horizontal rope produced in B.

Then since the sides of the triangle ABC are in the directions of the three forces, ABC may be taken as the triangle of forces.

Therefore
$$\frac{\mathbf{Q}}{BC}=\frac{\mathbf{P}}{CA}=\frac{112}{AB}.$$
$$\frac{\mathbf{Q}}{1}=\frac{\mathbf{P}}{2}=\frac{112}{\sqrt{3}}.$$
$$\mathbf{P}=\frac{224}{\sqrt{3}}=129{\cdot}3.$$
$$\mathbf{Q}=\frac{112}{\sqrt{3}}=64{\cdot}7.$$

The required tensions are **129·3 and 64·7 lbs.' wt.** correct to the first decimal place.

Example.—If the greatest tension which a picture wire can sustain without breaking is **80 lbs.' wt.**, find the greatest weight of a picture which the wire can just carry when it is attached to two rings in the ordinary way, and passed over a nail, each part of it making an angle of 60° with the vertical.

Let ABC denote the string supporting the picture.

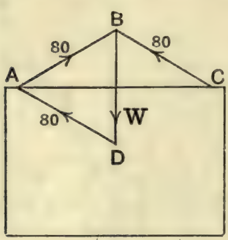

Fig. 20.

The string being on the point of breaking the tension in each part of it is 80 lbs.' wt.

Thus the picture is in equilibrium under the action of 80 lbs. wt. along AB, 80 lbs.' wt. along CB, and its own weight vertically downwards. Let this be **W** lbs.' wt.

Draw AD parallel to CB to meet the vertical through B in D.

Then ABD is the triangle of forces for the forces acting on the picture.

But since AB, AD are both inclined at 60° to BD, ABD is an equilateral triangle, *i.e.*

$$BD = DA = AB.$$

But

$$\frac{\mathbf{W}}{BD} = \frac{80}{DA} = \frac{80}{AB}.$$

$$\therefore \ \mathbf{W} = 80.$$

Thus the required weight of the picture **is 80 lbs.' wt.**

In this question we have assumed that the tensions in the two parts of the strings are equal. This follows from symmetry, because the arrangement is symmetrical, and there is no reason why the tension on one side should be greater than that on the other.

This, however, admits of exact proof by means of the triangle of forces, and is left as an exercise. See Exercises II. a. 2.

Example.—Two strings are tied to a post and pulled with tensions of 15 and 20 lbs.' wt., being inclined to each other at an angle of 45°. What is the entire pull on the post?

The required pull is the resultant of 15 and 20 lbs.' wt. acting in directions making an angle 45° with each other.

Draw AB, BC to represent the two pulls in direction and magnitude. Then AC represents the resultant.

Draw CD perpendicular to AB produced.

$$AC^2 = AB^2 + BC^2 + 2AB \cdot BD.$$

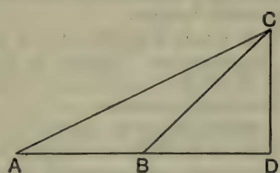

Fig. 21.

[Now we may suppose AB, BC to contain 15 and 20 units of length respectively; then AC contains as many units of length as the required force contains lbs.' wt.]

And

$$BD = \frac{1}{\sqrt{2}} \cdot BC;$$

$$\therefore \ AC^2 = AB^2 + BC^2 + \sqrt{2} \cdot AB \cdot BC.$$

Therefore if **R** lbs.' wt. is the required pull,

$$\mathbf{R}^2 = 15^2 + 20^2 + \sqrt{2} \, . \, 15 \, . \, 20$$
$$= 1049.$$
$$\mathbf{R} = 32{\cdot}4.$$

The entire pull on the post is **32·4 lbs.' wt.**

The note in brackets [ ] is given for explanation. It would not be necessary in a formal solution of the question.

Exercises I. c.

1. A 10 lb. weight is supported by two strings, one of which is inclined at 45° to the vertical and the other is horizontal. What are the tensions ?

2. A body hangs by a string and is pulled by a spring balance till the string makes an angle of 60° with the vertical. The balance then indicates 36 lbs.' wt. What is the weight of the body and the tension in the string ?

3. Find the resultant of two forces of 10 and 12 grams' weight inclined at 135°.

4. Find the resultant of two forces of 10 and 12 grams' weight inclined at 45°.

Summary.

Action of forces on body free to move. If two forces act together on a body quite free to move, the body will in general begin to move in some manner.

No result is produced only when the forces are **equal** and **opposite**.

Equilibrium. Forces, or the body on which they act, are said to be in equilibrium when the forces balance each other and produce no tendency to motion.

Measurement of forces. A force can be measured in terms of a unit, as for instance a pound's weight or a gram's weight.

Graphic representation of a force. A straight line can be drawn to represent a force (1) in direction, (2) in magnitude, with a chosen scale, (3) in line of action, (4) in sense, by adding an arrow-head.

Parallelogram of forces. If two forces are represented by the sides *AB*, *AD* of a parallelogram *ABCD*, their resultant is represented by the diagonal *AD*.

Measurement of angles. Angles are measured in degrees.

Triangle of forces. If three forces acting at a point can be represented by the sides of a triangle taken in order, the forces are in equilibrium.

The converse of this is also true.

CHAPTER II.

RESOLUTION OF FORCES. POLYGON OF FORCES.

Composition and Resolution of Forces.—To find the resultant of two given forces, or the single force to which they are equivalent, is called *compounding* the given forces.

To find two forces to which a given one is equivalent, or which would produce the same effect as the given force, is called *resolving* the given force.

The two forces thus found are called **components** of the given one.

Two given forces can only be compounded in one way, but one force may be resolved into two in an endless number of ways.

If OA represents *completely* a force, and we draw any parallelogram, such as $OBAC$, the forces represented by OB and OC, since they have OA for resultant, would produce the same effect as OA. Thus OA may be resolved into OB and OC.

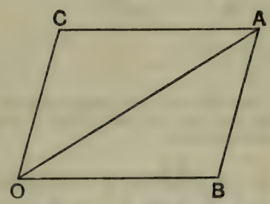

FIG. 22.—Components of a given force.

Resolution of Forces.—It is clear that a given force may be resolved into two forces along any two straight lines meeting at some point in its line of action, if the force lies in the same plane as these two lines. If OB, OC are the lines, we have to make OA to represent the given force, and draw AC, AB parallel to the given lines so as to form a parallelogram. Then OB, OC represent forces into which the given one may be resolved.

Again, to use the triangle, let AB represent a force in direction and magnitude. Then if P is any point whatever, AB is equivalent to, or may be resolved into, two forces represented in direction and magnitude by AP and PB.

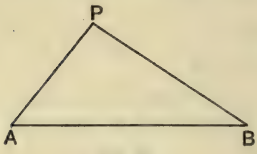

FIG. 23.

We may resolve a force into two having given directions, that is, parallel to given straight lines, by means of the triangle.

Let AB denote the given force in direction and magnitude. Let L and M be the given straight lines. Through A and B draw straight lines parallel to L and M. This may be done in two ways, so as to get either the triangle ACB or the triangle ADB; and the required forces are represented in direction and magnitude by CB and AC or by AD and DB. It is clear that these results are the same.

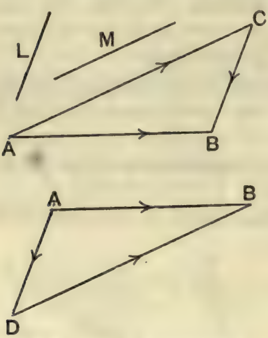

FIG. 24.—To find components for a given force parallel to two given straight lines.

The two forces obtained may then be supposed to act through any point on the line of action of the given force.

It is obvious from the construction that the components found for AB parallel to L and M have definite senses, and cannot have any others. The question would be impossible if the senses of the components parallel to L and M were assigned, and were different from those found; that is, in the case of our figure, the force parallel to L must be downward and to the left, and that parallel to M must be upward and to the right.

An Illustration.—To illustrate this; suppose it is required to sustain a weight by means of strings acting parallel to two given lines in the same vertical plane. This can always be done. The tensions in the strings have to be so adjusted as to produce a resultant vertically upwards and equal to the weight of the body.

Let L and M be the given lines. Draw AB vertically upwards and to represent the weight of the body. Construct the triangle ABC to give the components AC, CB of the force AB.

The figures show two different cases of drawing the lines L and M. But however they are drawn the problem is possible.

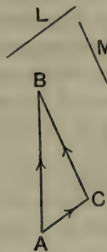

FIG. 25.

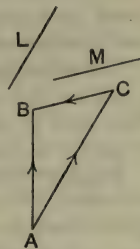

FIG. 26.

It is clear that the senses of the forces acting along the strings could not, in the case of either figure, be different from those that have been found, for we could not then construct a triangle to give AB as the resultant of the forces parallel to L and M. The same thing may also be inferred from practical experience, To take the second figure for example ; we could not have the pulls both upwards, for then they would be both to the right. as in Fig. 27 ; nor could we have them both downwards, as in

FIG. 27.

FIG. 28.

FIG. 29.

FIG. 30.

Fig. 28 ; nor could we have that parallel to L downwards, and that parallel to M upwards, as in Fig. 29. In none of these cases could the strings combine to produce an upward pull. The only way in which they can act is as in Fig. 30, which has been indicated in the construction by means of the triangle.

Rules for the Position of Forces.—The following rules for the position of the forces to be found follow from the use of the parallelogram or triangle, and agree with practical experience:

If three lines are drawn out of one point in the directions and senses of two forces and their resultant, that for the resultant must *lie in the angle between the other two, which is less than two right angles.*

If three lines are drawn out of one point in the directions and senses of three forces in equilibrium, they must form three angles each less than two right angles (or angles such as those in Euclid).

It is important to attend carefully to these points about the positions and senses of forces in solving questions, else much time may be spent in drawing impossible positions and trying to apply the triangle to them.

Another Condition.—The condition that two forces and their resultant or two forces and their equilibrant must lie in one plane must be noticed. This is implied in the fact that they can be represented by the sides of a triangle, because the three sides of a triangle must lie in one plane. Or, it follows from the fact that the sides and diagonal of a parallelogram are all in one plane. Thus, if two forces have their lines of action in the plane of this page, neither the single force to which they are equivalent nor the force which holds them in equilibrium can rise up above the plane of the page or pass down below it, but must also lie in the plane of the page.

Again, a weight can only be sustained by two strings if the strings are in one and the same vertical plane, so that the same plane may contain the two strings and the vertical line of action of the weight.

The reason of this may be seen in a general way as follows. If two of the forces act in one plane they have no tendency to produce or to prevent motion away from the plane either one way or the other, but only in the plane. If then the third force passes to one side of the plane it tends to produce motion towards that side ; and, there being nothing to prevent it, such motion will take place, so that there can be no equilibrium.

Action of Smooth Joints.—If a rod is loosely jointed the joint can only exert on it a simple pull or push, and cannot exert a twist as a stiff joint or hinge could do.

If a body is called 'light' in a question, it is to be understood as a rule that its weight is to be neglected, or is negligible as compared with the other forces in action.

If a light rod is in equilibrium under the action of two forces due to loose joints at its ends, these forces must be equal and oppositely directed along the same straight line. Thus they must both act along the line of the rod.

Now there are two ways in which these forces may act.

(1) Each joint may exert a pull along the rod away from the other joint, as indicated by the forces **F**, **F**′ in Fig. 31.

FIG. 31.—Actions of joints on rod.

The rod then exerts a *pull* along its length on each joint, and on any body connected with it. It is in a state of *tension*, as a string could be.

(2) Each joint may exert a push along the rod towards the other joint, as indicated by the forces **F**, **F**′ in Fig. 32.

FIG. 32.—Actions of joints on rod.

The rod then exerts a *push* or *thrust* along its length on each joint, and on any body connected with it. It is in a state of *compression*, as a string could never be.

The force acting along the rod, considered as acting on the rod itself, is sometimes spoken of as the *stress* in the rod. A stress of this sort may be a tension or a compression. The only stress which can exist in a string (such as we suppose in mechanical questions, entirely without stiffness) is a tension.

EXAMPLE.—*AB*, *AC* are two light rods loosely jointed together at *A* and to fixtures at *B* and *C*. *B* is above *A* and *C* is below *A*, and *AB*, *AC* are inclined at 30° and 60° to the horizon. A weight of 100 lbs. is hung on the joint at *A*. Find the natures and magnitudes of the forces in the rods.

In any question of this sort all the lines along which the forces act must be in some one plane (in this case a vertical plane), and we suppose that they are put into the plane of the paper.

To solve the question we consider the forces acting at the joint *A*, these being the weight of the 100 lbs. and the forces along the two rods, and get a triangle with its sides in the directions of these three, making what use we can of the lines of the figure representing the arrangement.

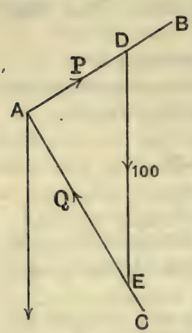

FIG. 33.

This may be done in several ways. If we draw the vertical DE, then DEA is the triangle of forces. DE represents the weight of the 100 lbs. acting vertically downwards; and AD, EA, representing the forces due to the rods, show that these are a pull along AB and a push along CA.

The triangle might also have been obtained by drawing from a point on the vertical through A a straight line parallel to either rod to meet the other one.

These triangles should both be drawn for exercise, and it should be made out that they both lead to the same result as we have just obtained.

The formal solution follows.

Let **P** and **Q** lbs.' wt. be the forces in the rods

Draw the vertical DE.

Then ADE is the triangle of forces.

And ADE is a right-angled △ having the angle ADE 60°.

$$\frac{\mathbf{P}}{AD}=\frac{\mathbf{Q}}{EA}=\frac{100}{DE}.$$

$$\frac{\mathbf{P}}{1}=\frac{\mathbf{Q}}{\sqrt{3}}=\frac{100}{2}.$$

$$\mathbf{P}=50;\ \mathbf{Q}=86{\cdot}6.$$

The required forces are

a tension in AB of 50 lbs.' wt.

a thrust in AC of 86·6 lbs.' wt.

The Sides of the Triangle are taken in Order.—In the above example note particularly that the way that we know that the forces on A due to the rods are a pull along AB and a push along CA is this. The forces are represented by the sides of DEA *taken in order*, or *the same way round.*

Now, since the force represented by the vertical side is downwards, this fixes the way of going round the triangle; therefore the senses of the others are from E to A and from A to D.

In writing down the lines which represent the forces, it is best always to write them in the senses of the forces, as has been done above in the calculation. Thus they are called AD, EA, DE.

In a case of this sort what is the body on which the three forces are acting? We may consider it to be the joint or peg or whatever connects the rods at A. This is pulled vertically downwards by the string which ties up the weight, with a force equal to 100 lbs.' wt., and it is also acted upon by the forces due to the rods. But we may just as well consider the forces to be acting on the 100 lbs. This is pulled vertically downwards by its own weight directly, and it is acted upon by the forces due to the rods, although not directly; yet these are transmitted to it somehow, and produce their effect in keeping it in equilibrium.

EXAMPLE.—A 50 gram weight hangs at the end of a string from a fixed point A. Another string is tied to the first 100 centimetres below A. Find the least tension in this second string which will hold the weight in a position 20 centimetres above its first position in which the first string is vertical.

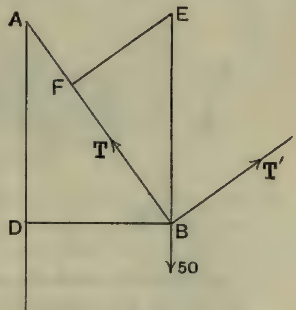

FIG. 34.

Let B be the junction of the two strings. Draw BD horizontally to meet the vertical through A in D.

In first position B was 100 cms. below A, and is now 80 cms. below A. So that $AD = 80$ cms.

$BD = \sqrt{(100)^2 - (80)^2}$ cms.
$= 60$ cms.

Draw BE vertically upwards to denote a force of 50 grams' wt., and EF parallel to the second string.

The body is in equilibrium under the action of 50 grams' wt. vertically downwards, and the tensions, say **T** and **T′** grams' wt. in the strings. *EBF* may be taken as triangle of forces.

Now, since **T** is to be as small as possible, *EF* is to be as short as possible; and this will be when it is perpendicular to *BA*.

The triangle *EBF*, then, has its angles equal to those of *BAD*; and is therefore similar to *BAD*, and the sides of the two triangles are proportional.

$$\frac{\mathbf{T}}{FE}=\frac{50}{EB}.$$

$$\therefore \frac{\mathbf{T}}{DB}=\frac{50}{BA}.$$

Or $$\frac{\mathbf{T}}{60}=\frac{50}{100},$$

$$\mathbf{T}=30.$$

∴ the required least tension is **30 grams' wt.**

It often happens, as in this example, that the triangle obtained as triangle of forces can be shown to be similar to some other triangle whose sides are known.

Example.—A weight is suspended by two strings at right angles to each other from two points on the same horizontal line. Show that the tension in each string is proportional to the length of the other.

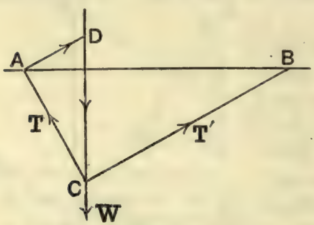

Fig. 35.

Let *AC*, *BC* be the strings, and let the tensions in them be **T**, **T′** units respectively, and the weight of the body **W** units.

Draw *AD* parallel to *CB* to meet the vertical through *C* in *D*. Then *DCA* is triangle of forces.

$$\therefore \frac{\mathbf{T}}{CA}=\frac{\mathbf{T}'}{AD}.$$

Now, $\triangle$ CAD is similar to triangle BCA;

$\because$ they have rt. $\angle^s$ at C and A,

and $\angle ACD$ = complement of $DCB = \angle ABC$.

$$\therefore \; CA : AD = BC : CA.$$

$$\therefore \; \frac{\mathbf{T}}{BC} = \frac{\mathbf{T}'}{CA};$$

i.e. tensions in CA, CB are proportional respectively to the lengths of CB, CA.

Jib and Tie.—This arrangement, employed in the crane for lifting heavy bodies, forms a very good illustration of the

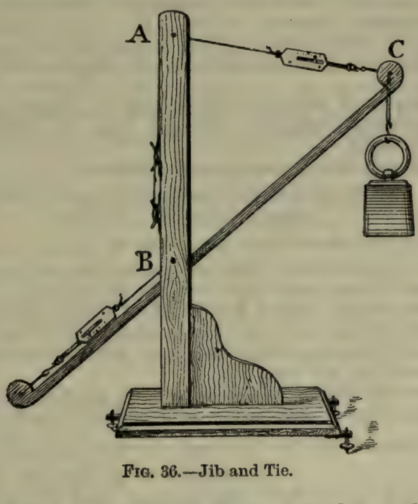

FIG. 36.—Jib and Tie.

triangle of forces. The figure shows a model of the arrangement. AB is a fixed vertical post, AC a string (or rope or chain), and BC a rigid rod. If a weight is suspended from C, the tension in the string and the stress in the rod, which is a thrust, act together to sustain the weight.

Suppose that the weights of the string and rod may be neglected in comparison with the weight of the body sustained. Let the weight, tension in string, and stress in rod be **W**, **T**, **P** units of any sort, respectively. ABC is the triangle of forces, and we have

$$\frac{\mathbf{W}}{AB}=\frac{\mathbf{P}}{BC}=\frac{\mathbf{T}}{CA}.$$

In the apparatus shown there are two spring balances to give the magnitudes of the forces in AC, BC, and a body of known weight is hung on at C. The readings of the balances and the weight may be compared with the corresponding lengths, and thus practical demonstrations may be obtained of the truth of the proposition for particular cases.

EXPERIMENT 6.—Take a light rod AB. Make a hole through the end A, and fasten the rod by means of a nail through this hole against the wall or other vertical support, so that it can turn freely in a vertical plane. Connect the end B to a point C vertically above A by means of a rubber band having strings tied to its ends.

Hang a weight from B and observe the length of the band when stretched.

Remove the band and find what weight is necessary to stretch it to the same length hanging vertically from it. This gives the tension in the band.

By the triangle of forces we have the relation

$$\frac{\text{Weight of body hung on } B}{\text{Tension in band}}=\frac{AC}{CB}.$$

These two ratios should then be compared.

The band is stretched a little to support the rod in its position, but if the rod is made very light this can be neglected. Or the rod can be taken so long that A is its middle point. Then it balances itself, and all the tension in the band is due to the weight at B.

The nail or pivot supporting the rod at A must be very small and work very easily in the hole.

The same experiment may be done with a spring balance in BC instead of the band. Then larger weights can be used.

EXAMPLE.—If in the jib and tie arrangement the jib is inclined at 45° and the tie at 60° to the vertical, find the relations between the stresses in them and the weight of the body sustained.

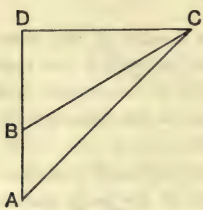

FIG. 37.

Let AC, BC denote the jib and the tie. Then if $\mathbf{P}$ and $\mathbf{Q}$ are the stresses in them, and $\mathbf{W}$ the weight;

$$\frac{\mathbf{P}}{AC}=\frac{\mathbf{Q}}{CB}=\frac{\mathbf{W}}{BA}.$$

We have, therefore, to compare the sides of the triangle ABC.

Draw CD horizontally to meet the vertical AB in D.

Call the length of BD 1.

Then $BC=2$, $DC=\sqrt{3}$, $DA=\sqrt{3}$, $AC=\sqrt{3}.\sqrt{2}=\sqrt{6}$.

$$\therefore\ AC : CB : BA = \sqrt{6} : 2 : \sqrt{3}-1.$$

$$\therefore\ \frac{\mathbf{P}}{\sqrt{6}}=\frac{\mathbf{Q}}{2}=\frac{\mathbf{W}}{\sqrt{3}-1}$$

$$\mathbf{P}=\frac{\sqrt{6}}{\sqrt{3}-1}.\mathbf{W}=\frac{\sqrt{6}(\sqrt{3}+1)}{2}\mathbf{W}.$$

$$\mathbf{Q}=\frac{2}{\sqrt{3}-1}.\mathbf{W}=(\sqrt{3}+1)\mathbf{W}.$$

This question affords an example of the use of the simple right-angled triangles, which should be carefully noticed. Having taken the sides of BCD as 1, 2, $\sqrt{3}$, as usual, we cannot then, of course, take those of ACD as 1, 1, $\sqrt{2}$; but we must call them $\sqrt{3}$, $\sqrt{3}$, $\sqrt{6}$, which have the same ratios.

Sailing Boat.—A good example of the resolution of forces is afforded by the action of the wind on a sailing boat. The figure is drawn to represent a horizontal plan. Let AB be the fore-and-aft line of the boat, the boat being set to go in the direction AB.

Let PO be the direction in which the wind is blowing, the figure showing the boat sailing against the wind.

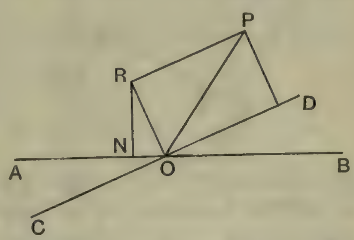

FIG. 38.—Showing action of wind on sails of boat.

Let COD be the line along which a sail is set.

The wind exerts on the sail a force which is mainly normal to the sail. Let RO represent this force.

RO is equivalent to two forces, represented in magnitude and direction by RN, NO.

NO is the force causing the forward motion of the boat.

Forces such as RN, due to the action of the wind on the sails and the components of pressure at right angles to AB of the wind on the hull and other parts of the boat, tend to cause motion at right angles to AB. This motion is called *lee-way*, but, as from the make of the boat it is much freer to move along than across AB, these forces do not produce much effect.

A boat in which the side forces produce little effect can sail with the angle POB small. This is called sailing *close to the wind.* Any boat can be *set* close to the wind; but it may happen that so much lee-way is made for a given amount of head-way that the actual line of progress is not close to the wind, and it may be more judicious to set the boat not quite so close.

EXAMPLE.—Two weights, $\mathbf{W}$, $\mathbf{W}'$ are attached to the points B, C of a string $ABCD$. AB is inclined at 30° to the vertical on one side, and BC, CD at 60° and 30° on the other side. Compare $\mathbf{W}$ and $\mathbf{W}'$.

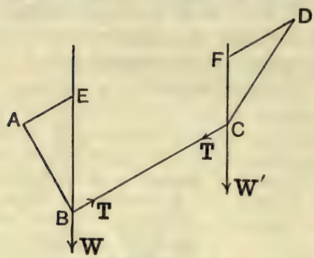

FIG. 39.

Let T be the tension in BC.

Draw AE, DF parallel to BC to meet the verticals through B and C in E and F. Then BAE, CDF are the triangles for the forces acting at B and C.

Thus, $$\frac{\mathbf{W}}{\mathbf{T}}=\frac{EB}{AE}=2;$$

and $$\frac{\mathbf{W}'}{\mathbf{T}}=\frac{FC}{DF}=1.$$

$$\therefore\ \mathbf{W}=2\mathbf{W}'.$$

EXAMPLE.—Two rafters AB, BC, each 20 feet long, are supported on the tops of two walls at A and C, these points being on the same level, and 32 feet apart. Find the additional horizontal pressures against the walls due to suspending a weight of 48 lbs. from B.

Join AC. Draw BED vertical, and AD parallel to BC. Then $AB=20'$, $AE=16'$, $BE=ED=12'$.

[NOTE.—The signs $'$ and $''$ are used to signify *feet* and *inches*. Thus $2'\ 6''$ means 2 feet 6 inches.]

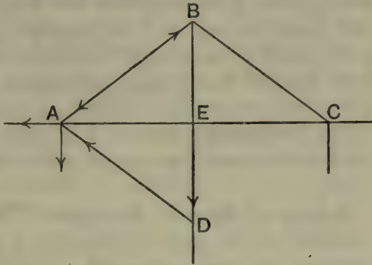

FIG. 40.

Let **R** lbs.' wt. be the stress produced in each rafter.

BDA is the triangle for the forces acting at B.

$$\therefore \frac{\mathbf{R}}{AB}=\frac{48}{BD};$$

$$\mathbf{R}=\frac{20}{24}\times 48=40.$$

Thus a thrust is produced on the top A of the wall in the direction BA. Let this be represented by BA. Then its horizontal and vertical components are represented by EA and BE.

$\therefore$ if the horizontal thrust produced on the wall is **P** lbs.' wt.,

$$\frac{\mathbf{P}}{EA}=\frac{40}{BA}.$$

$$\mathbf{P}=\frac{16}{20}\times 40=32.$$

$\therefore$ additional horizontal pressure on each wall is **32 lbs.' wt.**

In this question we are asked for the *additional* forces called into action. These will be independent of any forces already acting, and will be the same as if there were no forces acting before the 48 lbs. is hung on, that is, as if the rafters had no weight, and were supporting nothing else; and the question is solved in this way.

Actual rafters would, of course, be connected by a horizontal piece, a *tie rod*, the object of which is to take the horizontal pressures and relieve the walls. But nothing of the sort being mentioned, and the question distinctly implying that the walls take the pressures, we must suppose the tie rod not to exist.

The question has been done by finding the stress in the rafter *AB*, or the force acting along it. This at the end *A* is a force acting from *B* to *A* against the wall. Then we consider what two forces acting at *A* horizontally and vertically against the wall would be equivalent to this; that is, we have resolved the force into its horizontal and vertical components; and it is the horizontal component which is required.

The following question is similar in some respects. It involves the new idea of the action of *smooth surfaces* on each other.

Action of Smooth Surfaces.—When two perfectly smooth surfaces are in contact, or when a body is in contact with a perfectly smooth surface, there can be no action between the two bodies of such a nature as to prevent one from slipping over the other; that is, there can be no friction force, and the only action possible on either body in consequence of the contact is a force at right angles, or *normal*, to either surface at the point of contact. Thus, if a body rests on a smooth table, then, however it may be pulled or pushed, the table can only exert on it a force at right angles to its own surface; if the table is horizontal, and the body is upon it, this force must be vertically upwards. The body presses on the table, and the equal force with which the table presses back the body is sometimes spoken of as a *reaction*.

EXAMPLE.—Two rods *AB*, *BC*, without weight, each 20 inches long, are pivoted at *B*, and the ends *A*, *C* connected by a string 32 inches long. They are then placed with *A* and *C* in contact with a smooth horizontal table, the rods being in a vertical plane. A weight of 48 ounces is hung on the pivot at *B*. Find the tension in the string.

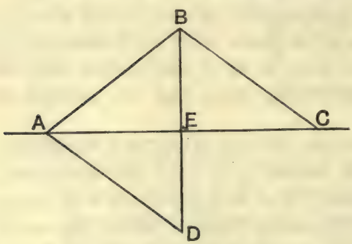

FIG. 41.

Let **T** ounces' wt. be the tension in the string, and **R** ounces' wt. the stress in each rod.

Draw BED vertical, and AD parallel to BC. Then

$$AE=16'', \quad BE=ED=12''.$$

ABD is the triangle for the forces acting at B.

$$\therefore \quad \frac{\mathbf{R}}{AB}=\frac{48}{BD}.$$

$$\mathbf{R}=\frac{20}{24}\times 48=40.$$

At the end A there are three forces in equilibrium, 40 ounces' wt. along BA, $\mathbf{T}$ ounces' wt. along AC, and the reaction of the table vertically upwards.

BAE is the triangle of forces.

$$\therefore \quad \frac{\mathbf{T}}{AE}=\frac{40}{BA}.$$

$$\mathbf{T}=\frac{16}{20}\times 40=32.$$

$\therefore$ the tension of the string is **32 ounces' wt.**

In this case we have considered the *equilibrium* of the three forces acting at A. If we had been asked for the horizontal pressure against a fixed support at A, it would have been more natural to resolve the thrust of 40 oz. wt. along BA and find its horizontal component, which is 32 oz. wt. along EA. In fact we may say that the component 32 oz. wt. along EA is the force acting against the string; and the string maintains equilibrium at A by exerting a force of 32 oz. wt. along AE.

This question can also be solved, more quickly, in the following way.

The table sustains the 48 oz. by upward pressures at A and B, and it is clear from the symmetry of the arrangement that these pressures must be equal.

Thus, considering the forces on the joint at A, these are the tension in the string along AE, the thrust in the rod BA along BA, and an upward force of 24 oz. wt.

AEB is the triangle for these forces.

$$\therefore \quad \frac{\mathbf{T}}{AE}=\frac{24}{EB}.$$

$$\mathbf{T}=\frac{16}{12}\times 24=32.$$

The required tension is **32 oz. wt.**

This solution involves the assumption that has been made about the pressures at A and B, which, although very simple, does not properly belong to the subject of this chapter.

A similar solution can be obtained for the last example about the rafters and their load.

The additional load of 48 lbs. at B causes an additional upward pressure of 24 lbs.' wt. from the wall at A, as well as an additional horizontal pressure from the wall along AC, and an additional thrust along AB. These three forces acting at A are in equilibrium, and we may use AEB as the triangle of forces, and thus find the relations between them.

Resultant of any number of Forces acting at a Point.—Let the four forces **1**, **2**, **3**, **4** act at the point O.

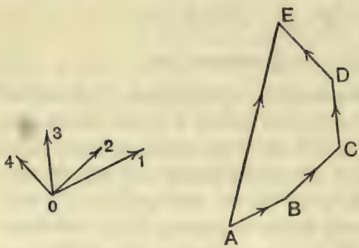

FIG. 42.—Resultant of several forces acting at a point.

Draw AB, BC to represent **1** and **2**, and so find their resultant AC.

1 and **2** may now be replaced by the force AC at O.

3 may be compounded with AC by drawing CD to represent **3**, and joining AD.

AD acting at O is the resultant of **1**, **2**, and **3**.

In the same way by drawing DE to represent **4**, we get AE as the resultant of AD and **4**, that is, as the resultant of **1, 2, 3, 4**.

It is clear that we can continue in the same way for any number of forces acting at one point.

In practice it is not necessary to draw the lines AC, AD, but only those which represent the given forces, each one starting from the point at which the last ended.

The lines representing the given forces may, of course, be drawn in any order, only care must be taken to draw each from the stopping point of the last in the proper sense for the corresponding force. The resultant finally obtained will, of course, act at O.

Condition for Equilibrium.—The given forces will be in equilibrium if, and only if, they have no resultant. For this the end of the line drawn to represent the last force must coincide with the starting point of the first. That is, in our figure, E must coincide with A. We may state then :

The necessary and sufficient condition for equilibrium of any number of forces acting at a point is that they can be represented in direction and magnitude by the sides, taken in order, of a closed polygon.

Polygon of Forces.—The proposition that this condition is sufficient for equilibrium, that is, that if the condition holds, the forces are in equilibrium, is called the *Polygon of Forces.* It is clearly an extension of the triangle of forces. We have here proved it and its converse.

It should be noted that it is not a necessary condition for any number of forces beyond three to be in equilibrium that they should act through one point; that is, they may in some cases be in equilibrium without all acting through one point.

EXPERIMENT 7.—Fasten three smooth pulleys to a vertical blackboard. Tie four strings together. Pass the ends of three over the pulleys, and let the fourth hang vertically down. Attach weights to the ends of the four strings, say 1, 1, 2, 3 pounds. Find the best position of rest for the knot O of the strings, as in Experiment 4. Draw lines OA, OB, OC, OD, marking the positions of the strings. Then forces 1, 1, 2, 3 along OA, OB, OC, OD are in equilibrium. Take a point a on the board and draw lines ab, bc, cd parallel to OA, OB, OC, and proportional to the forces 1, 1, 2. A scale of 6 ins. to a pound's weight will probably do in this case, but this, of course, depends on the size of the board. Join da. This line should be parallel to the line OD, and proportional to 3.

EXPERIMENT 8.—Do the same experiment with four rubber bands having strings attached, by stretching them out over drawing paper, and afterwards finding the weights necessary to produce the same extensions, as in Experiment 5.

Exercises II.

1. Show that if two forces are equal, their resultant bisects the angle between them.

2. If three forces are in equilibrium and two are equally inclined to the third, those two are equal.

3. Two horses walking along the sides of a canal pull a barge along by means of ropes, each inclined at 30° to the direction of motion. The tension in each rope is 120 lbs.' wt. What is the resistance of the water to the motion of the barge?

4. If a horse walking by the side of a canal draws a barge with a rope inclined at 30° to the canal, the tension in the rope being 120 lbs.' wt., find the resistance of the water to the forward motion of the barge and the side pressure which keeps the barge from going into the side.

[In this we have to find the two forces along and at right angles to the canal, which are counteracted by the pull of the rope.]

5. A weight hangs from a point A of a vertical wall. A rod BC is placed horizontally, with C fastened to a point of the string, and B vertically below A. The point B remaining fixed, show that the longer BC is, the greater will be both the thrust in the rod and the tension of the string AC.

6. In the above question the rod remaining of fixed length, show that, as it is brought lower down, the stress in it and the tension in the upper part of the string become less and less.

7. In the jib and tie arrangement, if the two members are inclined at 60° and 45° to the vertical, what are the stresses in them when a body weighing 1 cwt. is held up?

8. A weight W is sustained by two strings at 45° and 60° to the vertical. Show that the tensions in the strings are

$$\frac{\sqrt{6}}{\sqrt{3}+1} W \text{ and } \frac{2}{\sqrt{3}+1} W.$$

9. A weight W is sustained by two strings at 45° and 30° to the vertical. Show that the tensions in the strings are

$$\frac{\sqrt{2}}{\sqrt{3}+1} W \text{ and } \frac{2}{\sqrt{3}+1} W.$$

10. A kite and its appendages weigh 4 pounds. The string is inclined at 30° to the horizon, and has a tension of 4 lbs.' wt. Find the entire pressure of the wind against the kite, and the direction in which it acts.

11. A block of stone weighing 200 lbs. hangs at the end of a rope 20 feet long, the upper end being tied to a fixed support. Find the horizontal force and the least force which, when applied to the stone, will hold it 3 feet away from the vertical in which it lies.

12. A string $ABCD$ carries weights at B and C; AB makes 60° and CD 45° with the horizon; and BC is horizontal. If the weight at C is 100, what is the weight at A, and the tension in the part BC of the string?

13. Two rods AB, BC of equal length are loosely jointed at B, and the ends A and C are fixed at points on the same horizontal line by means of pins which would allow free rotation. The angle at B formed by the rods is turned upwards. A weight W is placed on B. Show that the stress in either rod is

$$\frac{W \cdot AB}{\sqrt{4AB^2 - AC^2}}.$$

14. Two light rods AB, BC, of lengths 4 and 3, are joined by a loose pin at B, and A and C are connected by a string of length 5. A and C rest on a smooth horizontal table, and B is vertically above AC. A weight W is hung on the pin. If T is the tension of the string, P and Q the stresses in the rods, and R and S the pressures on the table at A and B, show that

$$T=\frac{12W}{25},\ P=\frac{3W}{5},\ Q=\frac{4W}{5},\ R=\frac{9W}{25},\ S=\frac{16W}{25}.$$

15. ABC is a triangle, and in BC a point D is taken, such that $DC=2DB$. Show that, if forces acting at a point be represented by CA and $3AD$, then their resultant will be similarly represented by $2AB$. (Coll. Precep., 1898.)

16. One end of a string, whose length is 4 in., is fixed to a point in the surface of a solid uniform sphere of 5 in. diameter; the other end of the string is fixed to a smooth vertical wall, and the sphere is in equilibrium, hanging by the string and resting against the wall. The sphere weighs 24 lbs. Find the tension of the string and the pressure of the sphere on the wall. (Coll. Precep., 1898.)

17. Show how you would find, by means of a geometrical diagram, the resultant of any number of forces in one plane acting on a particle.

On a particle at O act three forces of 1 lb. wt., 2 lb. wt., $\frac{\sqrt{3}}{2}$ lb. wt. along the straight lines OA, OB, OC respectively. The angle AOB is 120°, and the angle BOC is 90°. Find the resultant of the forces in magnitude. (Coll. Precep., 1897.)

18. Show by a diagram drawn to scale the lines along which three forces of 13, 12, and 5 units must act if they are in equilibrium, and find from the diagram the angle between each pair of forces. (Science and Art, 1897.)

19. Draw lines AB, AC such that BAC is an angle of 70°; a force of 12 units acts from A to B, and a force of 15 units from A to C; find by construction their resultant, and if the resultant acts from A to D, find from your diagram the number of degrees in the angle BAD.

Specify the resultant. If the question had been, Find the force which will balance the two given forces, what difference would it make in the specification? (Science and Art, 1897.)

20. One end of a string is attached to a fixed point A, and after passing over a smooth peg B in the same horizontal plane, sustains a weight of P lbs.; a weight of 50 lbs. is now knotted to the string at C, midway between A and B. Find P, so that, in the position of equilibrium, AC may make an angle of 60° with AB. (Science and Art, 1898.)

21. If two forces act at a point, and their greatest and least resultants are represented by 22 and 2 respectively, find the forces. (Oxford Locals, 1896.)

22. What is meant by the component of a force in a given direction?

ABC is an equilateral triangle, *AD* is perpendicular to *BC*, and *AE* is parallel to *BC*. A force *P* acts along *AD*. What is its component (1) along *AB*, (2) along *AE*? (Oxford Locals, 1897.)

23. A mass of 4 lbs. is in equilibrium on a smooth horizontal plane when acted on by a horizontal force of 3 lbs.' wt. and by a force making an angle of 45° with the vertical drawn downwards and on the opposite side to that on which the horizontal force acts. Find the pressure exerted on the plane, and the magnitude of the force inclined to the vertical. (Camb. Jr. Loc., Mech., 1896.)

24. Explain the flight of a kite, and show that when it is at rest in the air the string cannot be at right angles to its plane. (Camb. Sr. Loc., Stat. Dyn. and Hydro., 1897.)

25. Find the smallest force which, when applied to the bob of a pendulum weighing 1 lb., will keep the string deflected through 30° from the vertical. Find also the tension of the string. (Camb. Sr. Loc., Stat. Dyn. and Hydro., 1898.)

Summary.

Resolution of a Force.—A force may, in general, be resolved, or broken up, into two other forces in directions parallel to any two given straight lines.

Action of a smooth Joint.—This is a simple force through the joint.

If a rod is acted on by the forces supplied by two smooth joints at its ends, and by no other forces, those two forces must be equal and oppositely directed along the rod.

Jib and Tie.—This arrangement is used in the crane. The actions in its parts are determined by means of the Triangle of Forces.

Sailing Boat.—By means of the figure and the resolution of forces we see that the action of the wind is to produce head-way and lee-way. The shape of the boat makes the lee-way small.

Action of a smooth Surface.—This is entirely at right angles to the surface.

Polygon of Forces.—If any number of forces acting at a point can be represented by the sides of a polygon taken in order, the forces are in equilibrium.

The converse of this is also true.

CHAPTER III.

ROTATIVE TENDENCY OF FORCE. MOMENTS.

A FORCE acting on a solid body in general tends to move it in two distinct ways:

(1) to *translate* it bodily from one place to another ;

(2) to *rotate* it so that its lines move into different directions.

The tendency of a force to produce translation we may take to be measured by the magnitude of the force.

Tendency to produce Rotation.—Let us now consider the tendency to produce rotation.

Take first the simplest case, in which the only motion of which the body is capable is one of simple rotation about a fixed point. The body may be supposed to be pierced with a hole as at O, and to have a peg passed through this hole and driven into a fixed support, so that the body is capable of turning freely about this peg. If a force like **F** now acts on the body, it is clear that it will tend to turn it about O in the counter-clockwise sense in the plane of the figure ; that is, in the sense opposite to that in which the hands of a clock move, and *will* so turn it unless it is acted upon by other forces to prevent this rotation.

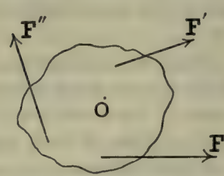

FIG. 43.—Rotative tendencies of forces.

If another force **F**′ now acts on the body and tends to rotate it in the clock-wise sense, and if **F** and **F**′ just counterbalance each other's effects, it is clear that we must consider the rotative tendencies of **F** and **F**′ about O to be equal and opposite. And if another force **F**″, when acting with **F**, just counterbalances the effect of **F**, we must consider the rotative tendencies of **F′** and **F″** about O to be equal and in the same, namely, the clockwise sense.

Measurement of Rotative Tendency.—What are we to take as the measure of the rotative tendency of one of these forces about O? This will depend on the relations that must hold among the forces, that is, among their magnitudes, positions, etc., in the case considered; that is, when **F** counterbalances **F′** or **F″**; and the question may be answered by experiment.

The required relations may be readily studied with a very simple piece of apparatus, consisting of a uniform bar 36 inches long, suspended at its middle point O, so that it is free to rotate, but will of itself rest horizontally, having no tendency of itself to rotate either way. It is marked in inches, and is notched along the upper edge above the inch marks. It is supplied with a set of weights which can be hung on it by means of the notches.

Now suppose, for example, that weights of 4 and 3 lbs. are hung on opposite sides of O, and adjusted so that the apparatus balances in the horizontal position. The rotative tendencies of these about O will then be equal. It is found that if the 4 lb. weight is 6 inches from O, the 3 lb. weight must be 8 inches from O; if the 4 is 12 inches off, the 3 must be 16 inches off, and so on, the product of each weight and its distance from O being the same in any case when the two weights balance each other.

Further experiments would show that two forces acting on a body free to turn about a fixed point will always balance each other if the forces tend to turn the body in opposite senses, and the product of each force and the perpendicular distance from the fixed point on it is the same. Thus, in the above example with the forces **F**, **F′**, **F″**, if n, n', n'' are the perpendiculars from the point O on the lines of action of the forces, the products $\mathbf{F}n$, $\mathbf{F}'n'$, $\mathbf{F}''n''$ must all be equal.

This result seems to indicate that the product $\mathbf{F}n$ may be taken as a measure of the twisting or rotative tendency of the force **F** about O. It would not do, however, to assume this yet; for although we know that when $\mathbf{F}n = \mathbf{F}'n'$ the two rotative tendencies are equal, it does not follow that they are equal or proportional to $\mathbf{F}n$ and $\mathbf{F}'n'$. [There may, for instance, be reasons for supposing that rotative tendency is proportional to the square of **F**n. The relation **F**n=**F**$'n'$ would still hold when the tendencies are equal.]

We may, however, proceed to reason as follows :

If we double the force **F**, keeping it in the same position as before, we may clearly assume that we have produced twice the rotative tendency about O. To balance this we must now double **F**$'n'$ by altering **F**$'$ or n' or both. Thus the rotative tendency of **F**$'$ is doubled by doubling the product **F**$'n'$.

Moment.—In the same manner it may be seen that the rotative tendency of **F**$'$ is increased in any ratio by increasing the product **F**$'n'$ in the same ratio. The rotative tendency is therefore proportional to this product, and the product may be taken as a *measure* of the tendency. It is called a **moment**, the formal definition of which is as follows :

The moment of a force about a point is the product of the force and the perpendicular from the point on the line of action of the force.

Further than this we may say that *the moment of a force about a point is the measure of the rotative tendency of the force about the point.*

Just as forces may be measured in terms of any suitable unit of force, so these perpendiculars may be measured in terms of any suitable unit of length. We obtain different measures for one and the same moment, or rotative tendency, according as we employ different units of force and length, but as long as we keep to same units in any one question, the measures obtained are consistent with each other.

Further experiments may be tried with the bar and weights to confirm the use of the moments as measures for rotative tendencies in cases in which several forces act so as to help each other. Suppose that several weights are hung on one side of O and produce equilibrium with one or several on the other side. Suppose, for example, that we have 4 lbs. at 3 ins. from O, and and 6 lbs. at 12 ins., both on the same side of O. The sum of the products, 4×3 and 6×12, is 84. And we should find that these could be balanced by 7 lbs. at 12 ins. from O on the other side, 7 times 12 being also 84. And in general we should find that there is equilibrium when the sum of the products of each weight by its distance from O taken on one side is equal to the similar sum taken on the other side.

Conditions for Equilibrium.—In general we should find that a body free to rotate about a fixed point remains in equilibrium if the sum of the moments of the forces tending to rotate it one way is equal to the sum of the moments of the forces tending to rotate it the other way; and whenever the body remains in equilibrium this condition holds.

This indicates that in a case of this sort as well, when there are more than two forces acting, we may look upon the moment of each force as the measure of its rotative tendency.

We may state concisely the conclusions at which we have arrived, thus:

The necessary and sufficient conditions for a body free to turn in a plane about a fixed point to be in equilibrium under the action of several forces acting on it in the plane are:

That the sum of the moments of the forces about the fixed point tending to turn the body one way should be equal to the sum of the moments of the forces about the same point tending to turn the body the other way.

Experiment 9.—Take a bar about 3 feet long, balanced at its middle point O, and divided off in inches to right and left of this point. Hang a 4 lb. weight at 15 inches to the left of O. Find what weights must be hung on to balance this at distances to the right of O of 3, 4, 5, 10, 12, 15 inches. Notice that the product of the mass in pounds and the number of inches is in every case the same as the corresponding product on the other side, that is 60.

Experiment 10.—With the same bar hang on a 1 lb. weight at 15 inches, and a 3 lb. weight at 5 inches to the left of O. Find all the possible positions in which an exact number of pounds can be hung at an exact number of inches to the right of O to balance these. Notice that the moment of the weight on the right is in every case equal to the sum of the moments of the weights on the left.

Experiment 11.—With the same bar hang 3 lbs. at 15 inches and 1 lb. at 10 inches to the left of O, and $1\frac{1}{2}$ lbs. at 10 inches to the right of O. Find what weights are necessary to maintain equilibrium at distances 4, 8, 10, 16 inches to right of O. Show that sum of moments of weights on left of O is equal to sum of moments of weights on right of O.

EXAMPLE.—A uniform rod is freely balanced at its middle point. Weights of 4 and 7 grams are hung at distances of 7 and 12 centimetres from the middle point on one side, and weights of 3, 2 and 1 grams at distances of 16, 20 and 12 centimetres on the other side. Where must a weight of 6 grams be placed to maintain equilibrium?

Sum of moments of 4 and 7 gram weights about middle point $=4\times7+7\times12=112$.

Sum of moments of 3, 2 and 1 gram weights about middle point $=3\times16+2\times20+1\times12=100$.

Hence the 6 gram weight must be hung on the side of the 3, 2 and 1.

It must be at such a distance as to produce a moment of $112-100$, or 12.

Thus it must be placed **at 2 centimetres from the middle point of the rod on the side of the 3, 2 and 1 gram weights.**

Exercises III. a.

Find the weights that must be hung at the given positions on a horizontal rod balanced on a point *O* so as to keep it in equilibrium in the following cases:

1. Rod loaded with 2 and 3 lbs. at 4 and 2 feet from *O* on same side: required weight to be 2 feet from *O* on the other side.

2. Rod loaded with 2 and 3 lbs. at 4 and 2 feet from *O* on opposite sides: required weight to be 2 feet from *O* on side of 2 lbs.

3. Find where a 3 lb. weight must be placed on the rod to balance it when it carries weights of 6 and 8 lbs. distant 4 and 3 inches from *O* on one side, and weights of 2 and 7 lbs. distant 5 and 6 inches on the other side.

Graphic Representation of a Moment.—To represent the moment of a force *F* about a point *O*, lay off a length *AB* along the line of action of **F**, to represent **F**. Draw the perpendicular *OL*.

Then the moment is represented by $AB\times OL$.

But $AB\times OL=$ twice area of triangle *OAB*.

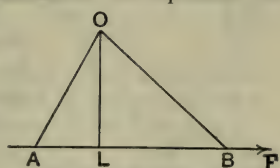

FIG. 44.—Graphic representation of moment of a force about a point.

∴ moment of **F** about *O* is represented by twice area of triangle *OAB*.

Algebraical Signs of Moments.—If two forces act on a body to turn it about a point, their resulting rotative tendency is the sum of their moments if they act to turn the body the same way; and the difference of their moments, and acts in the sense of the greater moment, if they act to turn the body in opposite ways. Hence, if we consider moments turning in one sense to be algebraically positive, and those turning in the other sense algebraically negative, the resulting rotative tendency of any number of forces is the *algebraic sum* of the given moments; and the sign of this sum will indicate in what sense it tends to rotate.

It is agreed to consider the counter-clockwise sense of rotation positive, and the clockwise sense negative in the signs of moments.

For instance, the algebraical sum of the moments about a point of weights 4 and 7 units, acting respectively 5 units of length to the left and 2 to the right of the point is

$$4 \times 5 - 7 \times 2 = 6 \text{ units of moment.}$$

We may now state the necessary and sufficient conditions of equilibrium of a body free to rotate about a fixed point as follows:

The algebraical sum of the moments of the forces acting on the body about the fixed point should be zero.

Effects of the Resultant.—The resultant of a given set of forces has the same effect as the given forces in all respects.

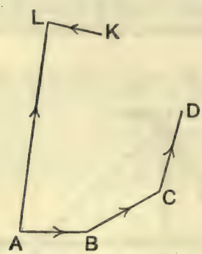

FIG. 45.—To find magnitude and direction of resultant.

(a) It has the same translative tendency.

(b) It has the same rotative tendency about any point.

(a) The direction and magnitude of the resultant of given forces is found by means of the polygon, and is represented by the side AL, completing the polygon whose sides AB, BC, CD, ... KL represent the given forces.

Any force represented by AL in direction and magnitude; or any force equal and parallel to the required resultant, and in the same sense, would have the same *translative tendency*.

This, however, tells us nothing about the position of the resultant.

(*b*) To have the same rotative tendency about any point the resultant must have a moment about the point equal to the algebraical sum of the moments of the given forces about the point.

Now the resultant has no moment about any point in its line of action, but has a moment about any other point. Hence the algebraical sum of the moments of the given forces about any point in the resultant is zero. And if we find any point, such that the algebraical sum of the moments of the given forces about it is zero, the resultant must pass through this point.

Having found the direction and magnitude of the resultant, this condition is sufficient to determine its position, and so to determine it completely.

Resultant of Two Parallel Forces.—These considerations enable us to determine very readily the resultant of two forces which are parallel to each other.

EXAMPLE.—To find the resultant of forces of 6 and 2 lbs.' weight, along parallel lines and in *the same* sense, acting at points A, B, and at right angles to the line AB, where $AB=4$ feet.

Since the resultant has the same translative tendency as the given forces, it must be 8 lbs.' weight, parallel to the given forces and in the same sense as these.

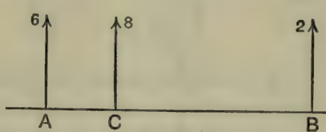

FIG. 46.

Let the resultant meet the line AB in C.

Then the given forces must have equal and opposite moments about C. Hence C must be between A and B. Let the distance AC be x feet; and BC be $4-x$ feet;

$$\therefore\quad 6\,.\,AC=2BC,$$

or

$$6x=2(4-x),$$

$$8x=8,$$

$$x=1.$$

Hence the resultant is **8 lbs.' wt. in direction and sense of given forces passing through a point distant 1 foot from A and 3 feet from B.**

Or, to find C, we may say, more simply :

Since resultant has same moment about *any* point as given forces taken together, we may use the point A.

Now moment of resultant about A is $8AC$ in positive sense.

Moment of force 6 about A is 0 ; and moment of force 2 about A is $2AB$, and is in positive sense.

$$\therefore \quad 8AC = 2AB$$

$$= 2 \times 4,$$

$$\therefore \quad AC = 1.$$

EXAMPLE.—To find the resultant of two forces of 6 and 2 lbs.' wt. along parallel lines and in *opposite* senses, acting at points A, B, and at right angles to AB, where $AB = 4$ feet.

By *translative principle*, resultant is 4 lbs.' wt. in direction of given forces, and in sense of force 6.

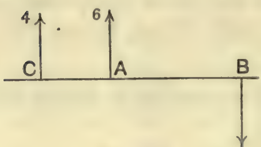

FIG. 47.

By *rotative principle*, if C in AB is a point on the line of action of the resultant, the forces 6 and 2 must have equal and opposite moments about C.

For the moments to be opposite, C must be on the line AB produced, towards one end or the other.

For the moments to be equal C must be nearer to the force 6 than to the force 2.

$$\therefore \quad C \text{ is on } BA \text{ produced towards } A.$$

Let $\qquad AC = x$ feet.

Then $\qquad 6x = 2(x+4).$

$$\therefore \quad x = 2 ;$$

$\therefore$ the resultant is **4 lbs.' wt. parallel to given forces, and in sense of the 6 lbs.' wt., passing through the point C on BA produced, where AC = 2 feet.**

Or, to find C, we may say more simply :

Since resultant has same moment about any point as given forces taken together, we may use the point A.

Moment of resultant about A is $4AC$, and is in negative sense.

Moment of force 6 about A is 0 ; and moment of force 2 about A is $2AB$, and is in negative sense.

$$\therefore \quad 4AC = 2AB$$
$$= 2 \times 4 ;$$
$$\therefore \quad AC = 2.$$

It may be noticed that the condition in these two examples, that AB should be at right angles to the lines of action of the given forces, was unnecessary. Because if this is not the case the products, such as $8AC$, $2AB$, used in the working, although they are not then the moments of the forces, are proportional to these moments, and may therefore be written in the equations instead of them, since by writing these instead of the true moments every term in the equation is changed in the same ratio. The result obtained in this case will then be the same as when AB is at right angles to the lines of action of all the forces.

This point will be shown more clearly a little later on.

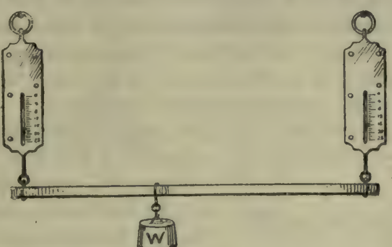

FIG. 48.—Three parallel forces in equilibrium.

EXPERIMENT 12.—Take a graduated bar, such as that used in the preceding experiments. Hang it up horizontally by means of two spring-balances, say at points 30 inches apart. The balances will be pulled out a little by the weight of the bar. Note carefully the indications.

Hang a weight of 5 lbs. at a point distant 12 ins. from one balance and 18 ins. from the other. Notice the *additional* indications of the balances. These additional indications are the measures of two forces which have the weight of the 5 lbs. for their equilibrant.

Note that the 5 lbs.' wt. is equal to the sum of these indications, and has the same moment about any point, say the point at which the left-hand balance is, as the two additional forces in the balances have.

EXPERIMENT 13.—Lay the bar down flat on a table, and pull it with two balances on one side and one on the other along parallel lines. Any positions may be taken. To start with, positions at 24 ins. apart may be used for the two balances on one side, and a point 8 ins. from one of these and 16 from the other for the third balance. Notice the indications carefully, and show that the proper conditions for three parallel forces in equilibrium are satisfied.

Two parallel forces acting in the same sense are called *like parallel forces.*

Two parallel forces acting in opposite senses are called *unlike parallel forces.*

Summary of Results.—It has already been indicated in examples how the resultant of two parallel forces may be found; and it is recommended always to employ this method in particular cases, making use of the two principles :

(*a*) *of translative tendency* to find the magnitude and direction (including sense) of the resultant ;

(*b*) *of rotative tendency* to find the line of action of the resultant.

General Formulae for Resultant.—The same method may be employed to find general formulae for the resultant, as will now be shown ; but it is preferable not to use the general formulae, but the principles, in working out examples.

To find the resultant of two like parallel forces **P** *and* **Q** *acting at points A and B.*

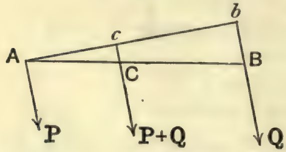

FIG. 49.—Composition of like parallel forces.

Since the translative tendency of the resultant must be equal to that of the two forces, the resultant must be **P**+**Q**, parallel to the forces and in the same sense as them.

This resultant must produce the same rotative tendency about any point as the two forces. Consider the point A.

Let the line of action of **P**+**Q** meet AB in C. Draw Acb at right angles to the common direction of the forces, meeting the lines of **Q** and **P**+**Q** in *b* and *c*.

Then the moments of **P**, **Q** and **P**+**Q** about *A* are

$$0,\ \mathbf{Q}\,.\,Ab,\ (\mathbf{P}+\mathbf{Q})\,Ac.$$

$$\therefore\ (\mathbf{P}+\mathbf{Q})\,Ac=\mathbf{Q}\,.\,Ab.$$

But
$$\frac{AC}{Ac}=\frac{AB}{Ab}.$$

$$\therefore\ (\mathbf{P}+\mathbf{Q})\,AC=\mathbf{Q}\,.\,AB.$$

$$\therefore\ AC=\frac{\mathbf{Q}}{\mathbf{P}+\mathbf{Q}}\,.\,AB.$$

Similarly we may show that

$$BC=\frac{\mathbf{P}}{\mathbf{P}+\mathbf{Q}}\,.\,AB.$$

Or we may get this value for *BC* by subtracting *AC* from *AB*.

To find the resultant of two unlike parallel forces **P** *and* **Q** *acting at points A and B.*

Let **P** be greater than **Q**.

By the principle of translative effect the resultant must be **P**−**Q** parallel to the given forces and in the sense of **P**.

P−**Q** must produce the same rotative effect about *A* as **Q** does, because the rotative effect of **P** about *A* is nothing.

P−**Q** must therefore meet *BA* produced ; let it be in *C*.

Draw *cAb* at right angles to the forces.

Then $(\mathbf{P}-\mathbf{Q})\,Ac=\mathbf{Q}\,.\,Ab.$

But
$$\frac{AC}{Ac}=\frac{AB}{Ab}.$$

$$\therefore\ (\mathbf{P}-\mathbf{Q})\,AC=\mathbf{Q}\,.\,AB.$$

$$AC=\frac{\mathbf{Q}}{\mathbf{P}-\mathbf{Q}}\,.\,AB.$$

Similarly we may show that

$$BC=\frac{\mathbf{P}}{\mathbf{P}-\mathbf{Q}}\,.\,AB.$$

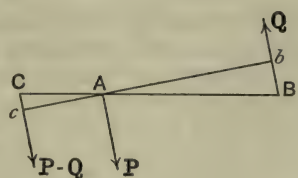

FIG. 50.—Composition of unlike parallel forces.

Or we may get this value for *BC* by adding *AC* to *AB*.

Equilibrant of Parallel Forces.—If we have to find the equilibrant of the parallel forces **P** and **Q** in either of these cases, it will of course be a force **P**+**Q** or **P**−**Q** acting through *C* opposite to the resultant that has been found.

Or in any particular case the equilibrant would be found by considering that it must produce exactly opposite tendencies, translative and rotative, to the given forces, or that it and the given forces produce no tendency to move a body on which they act at all.

If three parallel forces are in equilibrium, it is clear from the cases that have just been examined that the largest is in the middle and is opposite to the other two. This is a useful point to remember. In case the resultant of two parallel forces, either like or unlike, has to be found, if there is any difficulty about determining its position, consider first where the equilibrant must be so as to satisfy this condition about the relative positions; and then the resultant is exactly opposite to it.

Taking Moments.—When we write down an equation showing that the moment of a resultant about a point is equal to the sum of the moments of its components, or that the sum of the moments about a point of forces in equilibrium is zero, the operation is called *taking moments* about a point.

It is clear from the working given above that in taking moments for a set of parallel forces, which are all equally inclined to a given straight line, we may write down such products as **Q** . AB, etc., that is, the products of the forces by distances *along the line*; because although these are not the true moments they all have the same ratio to the corresponding true moments.

The magnitude and direction of the resultant of two parallel forces, as stated above, may be considered to be found by the parallelogram or triangle, this being a particular case of the use of the parallelogram or triangle.

Suppose, for example, we draw AB, BC to represent in direction and magnitude two like parallel forces. AC represents their resultant, by the triangle.

FIG. 51.

FIG. 52.

In this case the triangle ABC has its two sides AB, BC in one straight line: hence the third side coincides with them in direction and is equal to their sum.

Similarly, if in Fig. 52 AB, BC represent two unlike parallel forces, AC represents their resultant, coinciding with them in direction, being equal to their difference and in the sense of the greater.

EXAMPLE.—Two unlike parallel forces of 7 and 3 lbs.' wt. act at points A and B, 6 ins. apart; find their resultant.

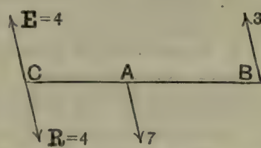

FIG. 53.—Resultant and equilibrant of two unlike parallel forces.

The resultant is a force of 4 lbs.' wt. parallel to the given forces, and in the sense of the 7 lbs.' wt. Also the resultant must act through a point C in BA produced towards the end A.

[This may be seen by considering that the equilibrant E, which is 4 lbs.' wt., must act with the 3, and the 7 must act against and between them.]

To find C, take moments about A.

[That is, we now express that the rotative tendency of R about A is equal to that of the 7 and 3 together about A, and in the same sense, that of the 7 at the same time being zero.]

Thus, $$4 \,.\, AC = 3 \times 6 \text{ ins.}$$
$$AC = 4\tfrac{1}{2} \text{ ins.}$$

Hence the resultant is **a force of 4 lbs.' wt. parallel to the given forces in the sense of the 7 lbs.' wt., and meeting the line BA produced at C, so that $AC = 4\frac{1}{2}$ inches.**

Notice that it is convenient in practice to take moments about a point through which one of the given forces acts, because that force has no moment about the point, and the equation of moments is consequently simplified. If we took moments, however, about any point whatever, we ought to obtain the same position for the resultant.

Exercises III. b.

Find the resultants of the following pairs of parallel forces acting at the points A and B respectively.

1. 2 and 3 lbs.' wt. in same sense. $AB=10$ feet.

2. 7 and 10 units in same sense. $AB=5$.

3. 7 and 10 units in opposite senses. $AB=5$.

4. 25 and 17 oz. wt. in opposite senses. $AB=4$ inches.

Find the equilibrants of the following pairs of parallel forces acting at the points A and B respectively.

5. 100 and 75 tons' wt. in opposite senses. $AB=4$ inches.

6. $5P$ and $3P$ in opposite senses. $AB=4a$.

7. 4 and 37 units in same sense. $AB=4$ inches.

8. 8 and 11 oz. wt. $AB=4$ ft. 9 ins.

9. A force of 10 units is balanced by one of 7 units acting in the opposite direction to itself and at a distance of 1 foot from it, and by another. What and where is the other?

10. The resultant of two like parallel forces, 7 and 10 oz. wt., is at a distance of 18 ins. from the greater. What is the distance between the given forces?

11. There are two unlike parallel forces $5P$ and $3P$. The distance of the resultant from the greater is $6a$. What is the distance between the forces?

12. A force P is resolved into two parallel components X, Y, at distances x, y from it on opposite sides. Find them.

13. A force P is resolved into two parallel components X, Y, at distances x, y from it on the same side, x being greater than y. Find them.

14. P, Q are two like parallel forces 1 foot apart; $3P=5Q$. Where is their resultant?

15. P, Q are two unlike parallel forces 1 foot apart; $4P=7Q$. Where is their resultant?

Summary.

A force acting on a body in general tends **to translate** and to **rotate** the body.

The rotative tendency of a force acting on a body free to turn about a fixed point is measured by the moment of the force about the point, which is the product of the force and the perpendicular distance from the point to the line of action of the force.

Condition of Equilibrium.—If a body is free to rotate about a fixed point, and several forces act on it, the necessary and sufficient condition for equilibrium is that the sum of the moments of the forces tending to turn one way should be equal to the sum of the moments of the forces tending to turn the other way.

Graphic representation of moment of a force about a point is twice area of triangle whose base is the line fully representing the force and vertex the given point.

Sign of Moment.—If a force tends to produce rotation about a point in the counter-clockwise sense, its moment about the point is reckoned positive ; if in the clock-wise sense, negative.

Condition for equilibrium may be stated thus : Algebraical sum of moments about fixed point must be zero.

The resultant of a given set of forces is the force which has

(*a*) the same translative tendency,

(*b*) the same rotative tendency about any point

as the given forces.

Thus, for a given set of parallel forces we may find :

(*a*) the magnitude, direction, and sense of the resultant by the translative principle ; for it must be parallel to the given forces, equal to the difference between the sum of those acting in one sense and the sum of those acting in the opposite sense, and in the sense of the greater sum ;

(*b*) the position of the resultant ; for it must produce the same rotative tendency about any point as all the forces taken together, and in the same sense of rotation.

CHAPTER IV.

PARALLEL FORCES. CENTRE OF PARALLEL FORCES. COUPLES.

We have now shown how simple questions about parallel forces and moments may be solved by the help of rules and principles deduced entirely from experiment.

We shall consider next some theoretical propositions about parallel forces and moments, and shall show that these lead to the same results. We shall begin with the propositions which determine fully the resultant of two parallel forces, just as the parallelogram of forces determines the resultant of two intersecting forces.

Resultant of two like parallel forces P and Q acting at points *A* and *B*.

Insert two forces, each equal to **S**, acting at *A* and *B* along *BA* produced and *AB* produced. These produce no effect on the system, for they neutralize each other.

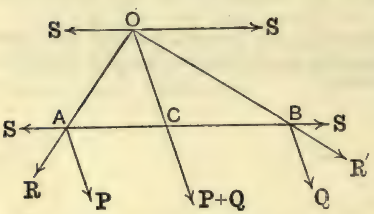

Fig. 54.—Composition of two like parallel forces.

Let **R** be the resultant of **P** and **S** at *A*, and **R′** the resultant of **Q** and **S** at *B*.

Let the lines of action of **R** and **R′** meet in *O*.

Then we may suppose **R** and **R′** to act at *O*.

Now resolve **R** into its components **P** and **S**, acting parallel to the original **P** and **S**; and similarly resolve **R**′ into its components **Q** and **S**.

The forces **S** and **S** neutralize each other; and we have left the single force **P**+**Q**.

Let C be the point in which the line of action of this force meets AB.

Since the triangle OCA has its sides in the directions of the forces **P** and **S** and their resultant **R**,

$$\therefore \frac{CA}{OC} = \frac{\mathbf{S}}{\mathbf{P}}.$$

Similarly from △ OCB,

$$\frac{CB}{OC} = \frac{\mathbf{S}}{\mathbf{Q}}.$$

$$\therefore \frac{CA}{CB} = \frac{\mathbf{Q}}{\mathbf{P}}.$$

And

$$CA + CB = AB.$$

$$\therefore CA = \frac{\mathbf{Q}}{\mathbf{P}+\mathbf{Q}} \cdot AB; \quad CB = \frac{\mathbf{P}}{\mathbf{P}+\mathbf{Q}} \cdot AB.$$

Resultant of two unlike parallel forces P and Q acting at points *A* and *B*.

Suppose **P** to be greater than **Q**.

Introduce forces **S**, **S** as before.

Transfer the resultants, **R** and **R**′, of **P** and **S** and of **Q** and **S** to O, the point in which the lines of these resultants meet.

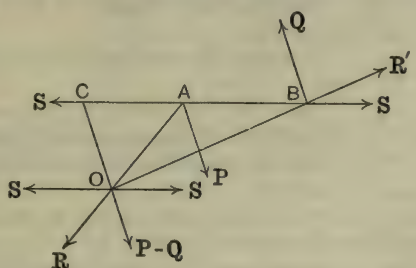

FIG. 55.—Composition of two unlike parallel forces.

Resolve **R** into its components **P** and **S** at O, and **R**′ into its components **Q** and **S** at O.

Thus we get the force **P** − **Q**.

Suppose its line of action meets the line AB in C.

By triangles, as before,

$$\frac{AC}{CO}=\frac{\mathbf{S}}{\mathbf{P}}.$$

$$\frac{CB}{OC}=\frac{\mathbf{S}}{\mathbf{Q}}.$$

$$\therefore \frac{AC}{CB}=\frac{\mathbf{Q}}{\mathbf{P}}.$$

And $$CB-AC=AB.$$

$$\therefore AC=\frac{\mathbf{Q}}{\mathbf{P}-\mathbf{Q}}\cdot AB;\quad CB=\frac{\mathbf{P}}{\mathbf{P}-\mathbf{Q}}\cdot AB.$$

To prove theoretically that

If a body is free to turn about a fixed point O, and is acted upon by two forces **P** *and* **Q**, *which hold it in equilibrium, the moments of* **P** *and* **Q** *about O are equal and in opposite senses.*

(i) Let the forces be intersecting.

Let the lines of action of **P** and **Q** meet in A.

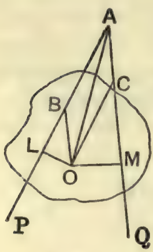

FIG. 56.

The action of the fixed support at O is to exert a force on the body whose line of action must pass through O; and, since this force with **P** and **Q** produces equilibrium, it must also act through A.

Thus the body is acted on by a third force along OA.

Therefore the resultant of **P** and **Q** is along AO.

Draw OB, OC parallel to the lines of **Q** and **P**, and OM, OL perpendicular to them.

Then **P**, **Q** and their resultant may be represented by AB, AC and AO.

The moments of **P** and **Q** about O are, on the same scale, represented by $AB \times OL$, and $AC \times OM$.

But $\triangle\ ABO=\frac{1}{2}AB \times OL$, and $\triangle\ ACO=\frac{1}{2}AC \times OM$, and

$$\triangle\ ABO=\triangle\ ACO.$$

$$\therefore AB \times OL=AC \times OM.$$

$$\therefore \mathbf{P} \times OL=\mathbf{Q} \times OM.$$

(ii) Let the forces be parallel.

Draw through O the straight line LM perpendicular to the lines of action of the forces, to intersect these lines in L and M.

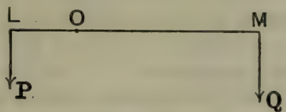

FIG. 57.

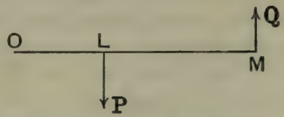

FIG. 58.

Then, as before, the resultant of **P** and **Q** must pass through O.

Hence O must be situated as in the figures, the first representing the case in which **P** and **Q** are like, and the second the case in which **P** and **Q** are unlike; and we must have the relation

$$\mathbf{P} \times OL = \mathbf{Q} \times OM.$$

Thus, in every case, the moments of **P** and **Q** about O are equal in magnitude and in opposite senses.

Conversely,

If a body is free to turn about a fixed point O, and is acted upon by two forces **P** *and* **Q**, *such that the moments of* **P** *and* **Q** *about O are equal and in opposite senses, the body is in equilibrium.*

(i) Let the forces be intersecting.

Make the same construction as before.

$$\therefore \quad \mathbf{P} \times OL = \mathbf{Q} \times OM,$$

and

$$AB \times OL = AC \times OM.$$

$$\therefore \quad \frac{\mathbf{P}}{AB} = \frac{\mathbf{Q}}{OC}.$$

$\therefore$ **P** and **Q** may be represented completely by AB and AC.

$\therefore$ the resultant of **P** and **Q** acts along AO, that is, through O.

(ii) Let the forces be parallel.

Make the same construction as before.

$\therefore$ $\mathbf{P} \times OL = \mathbf{Q} \times OM$, O being between **P** and **Q** if the forces are like, and outside them if they are unlike.

$\therefore$ the resultant of **P** and **Q** passes through O.

Now a force acting through O, the fixed point, has no tendency to turn the body one way or the other.

Therefore the body is in equilibrium.

THEORETICAL TREATMENT OF MOMENTS.

The conclusions at which we have arrived as the result of experiment about the moments of any number of forces may also be proved theoretically, starting with the parallelogram of forces as basis.

We shall begin with the following proposition :

The algebraical sum of the moments of two forces about any point in their plane is equal to the moment of their resultant about the same point.

(i) **Let the forces be intersecting.**

This depends on the following geometrical proposition.

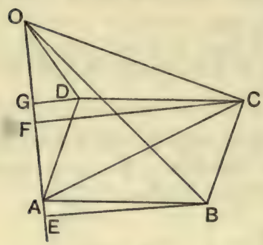

FIG. 59.

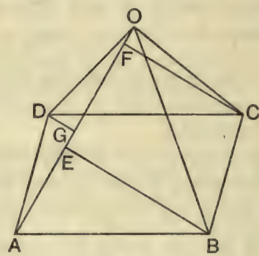

FIG. 60.

Let $ABCD$ be a parallelogram, and AC its diagonal, and O any point.

Then (1) if O is without the angle BAD and its vertically opposite angle, $\triangle\ OAC = \triangle\ OAB + \triangle\ OAD$;

(2) if O is within the angle BAD or its vertically opposite angle,

$$\triangle\ OAC = \text{difference between } \triangle^s\ OAB,\ OAD.$$

Draw BE, CF, DG perpendicular to OA.

Through B draw BH parallel to AO.

Then it is easily seen that $CH = DG$, and $HF = BE$.

$\therefore$ in Fig. 59 $CF = BE + DG$; in Fig. 60 $CF = BE - DG$.

$\therefore$ in case (1)

$$\tfrac{1}{2}AO\,.\,CF = \tfrac{1}{2}AO\,.\,BE + \tfrac{1}{2}AO\,.\,DG,$$

i.e. $$\triangle\ OAC = \triangle\ OAB + \triangle\ OAD.$$

In the same way in case (2), if O is in the angle DAC or its vertically opposite angle,

$$\triangle\, OAC = \triangle\, OAB - \triangle\, OAD;$$

and if O is in the angle CAB or its vertically opposite,

$$\triangle\, OAC = \triangle\, OAD - \triangle\, OAB.$$

Now let AB, AD represent completely two forces, and AC their resultant.

Then the moments of the three forces about O are represented numerically by twice the areas of the triangles OAB, OAD, OAC.

And by what has just been proved, if O is without the angle BAD and its vertically opposite angle, so that the moments of all three forces round O are in the same sense, the moment of AD about O is equal to the sum of the moments of AB and AD about O.

And if O lies within the angle BAD or its vertically opposite angle, so that the moments of AB and AD about O are in opposite senses, the moment of AC about O is equal to the difference of the moments of AB and AD about O, and is in the sense of the greater of these.

(ii) **Let the forces be parallel.**

Let **P** and **Q** be the forces and O the point.

Draw through O a straight line at right angles to the lines of the forces, meeting them in A and B.

Let the resultant **R** of **P** and **Q** meet this line in C.

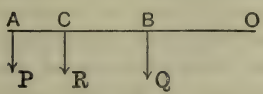

Fig. 61.

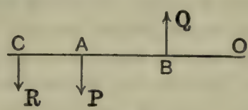

Fig. 62.

Suppose **P** and **Q** are like, and O is in AB produced.

Then $\mathbf{P} \,.\, OA + \mathbf{Q} \,.\, OB = (\mathbf{P}+\mathbf{Q})\, OC + \mathbf{P} \,.\, CA - \mathbf{Q} \,.\, CB$

$= \mathbf{R} \,.\, OC.$

$\because\ \mathbf{P}+\mathbf{Q} = \mathbf{R}$, and $\mathbf{P} \,.\, CA = \mathbf{Q} \,.\, CB.$

Suppose **P** and **Q** are unlike, and O is in AB produced.

Then $\mathbf{P} \,.\, OA - \mathbf{Q} \,.\, OB = (\mathbf{P}-\mathbf{Q})\, OC - \mathbf{P} \,.\, CA + \mathbf{Q} \,.\, CB$

$= \mathbf{R} \,.\, OC.$

In a similar manner the cases for all other positions of O, whether **P** and **Q** are like or unlike, could be proved.

Thus, in general, the algebraical sum of the moments of the given forces about O = the moment of the resultant about O.

The Case of any Number of Forces.—This proposition may now be extended to the case of any number of forces.

The algebraical sum of the moments of any number of forces about a point in their plane, is equal to the moment of their resultant about that point.

Call the forces **A**, **B**, **C**, **D**, etc.

Sum of moments of **A** and **B** = moment of their resultant.

Add moment of **C**.

Then sum of moments of this resultant and **C** = moment of *their* resultant.

Add moment of **D**.

Then sum of moments of this resultant and **D** = moment of their resultant.

And so on.

Thus the sum of the moments of all the forces = the moment of final resultant.

But this final resultant, got by finding resultant of **A** and **B**, then resultant of this force and **C**, then resultant of this and **D**, and so on, is the resultant of all the forces.

Hence the proposition.

If a body has a point fixed, about which it is free to rotate, it will be in equilibrium only when its resultant acts through this point, for then only does the resultant produce no tendency to rotate the body either way.

Thus the body is in equilibrium only when the moment of the resultant about the fixed point is zero, that is, only when the algebraical sum of the moments of the given forces about the fixed point is zero.

Therefore *the necessary and sufficient condition for equilibrium of a body free to rotate about a fixed point is:*
that the algebraical sum of the moments of the forces acting on it about the fixed point should be zero.

Recapitulation.—Notice that this result has already been stated on purely experimental grounds. It has now been proved theoretically by assuming two principles that rest on experimental evidence :

(1) The parallelogram of forces ;

(2) The fact that a body free to turn about a fixed point and acted upon by a single force is in equilibrium when, and only when, the force acts through the fixed point.

The second of these may be reduced to another principle, already stated, that two forces can only neutralize each other if they are equal and oppositely directed along the same straight line. For the fixed point about which the body is quite free to turn can only act on the body with a force passing through itself. Therefore, for the two forces to be in a straight line, the other force must also pass through the fixed point.

These theoretical results indicate that the moment of a force about a point may be used as a measure of the rotative tendency of the force about the point ; a result already arrived at mainly by means of experiment, and by assuming that the rotative tendency of a force about a point is proportional to the magnitude of the force as long as its position remains unchanged.

In the same way as in the case of two parallel forces, the resultant or equilibrant of any number of given parallel forces can be readily found.

(*a*) By the translative principle we find the magnitude and direction.

(*b*) By the rotative principle we find the position.

We must be careful to note that if the *resultant* is required the translative and rotative tendencies are *the same in sense* as the tendencies of the given forces acting together ; if the *equilibrant* is required, the translative and rotative tendencies must be *opposite in sense* to the tendencies of the given forces.

This is illustrated in the following example.

> Example.—A, B, C are points in order along a light rigid rod. $AB=2$ ins., $BC=3$ ins. Weights of 5 and 6 lbs. are hung at A and C. A string passes vertically upwards from B over a smooth pulley, and from its end a weight of 4 lbs. is hung. Find the force that will keep the rod at rest.

Draw the diagram to represent the arrangement.

The force necessary to keep the rod at rest is one of $6+5-4$, that is, 7 lbs.' wt. vertically upwards.

Consider the moments about A.

Of the given forces the sum of the moments in the negative sense is $6 \times 5 = 30$;

the sum of the moments in the positive sense is $4 \times 2 = 8$.

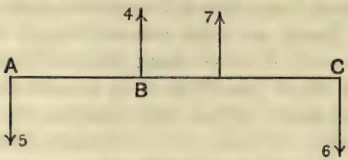

FIG. 63.

Thus the given forces tend to turn round A in the negative sense.

The force 7 lbs.' wt. must tend to turn round A in the positive sense; therefore it must act to the right of A.

Let its distance to the right of A be x ins.

Then by moments about A,

$$7x = 6 \cdot 5 - 4 \cdot 2 = 22,$$

$$x = 3\tfrac{1}{7}.$$

The required force is one of **7 lbs.' wt. vertically upwards from a point distant $3\frac{1}{7}$ ins. from A towards C.**

The same results would, of course, be obtained by taking moments about any other point.

Suppose, for instance, that the force of 7 lbs.' wt. acts at y ins. to the left of C. Then by moments about C we have

$$7y = 5 \cdot 5 - 4 \cdot 3 = 13,$$

$$y = 1\tfrac{6}{7}.$$

This agrees with the result already found.

Checking Results.—To test the accuracy of the position found for a resultant or an equilibrant in a case of this sort it is a good plan to take some other point, different from the one about which the moments have been taken, and see whether the moment of the force found about this new point is equal to the difference between the moments of the given forces about the same point in one sense and those in the other sense, that is, to the algebraical sum of the moments of the given forces; the force found having a moment in the same sense as this sum, or the opposite sense according as it is resultant or equilibrant.

Thus, in the above example, having found that the required force acts at $3\frac{1}{7}$ ins. from A, this is $1\frac{1}{7}$ ins. from B.

Moment of the force about B is $7 \cdot 1\frac{1}{7} = 8$.

Algebraical sum of moments of given forces about B $= 5 \cdot 2 - 6 \cdot 3 = -8$.

Thus, force found has equal and opposite moment about B to that of the given set of forces.

Experiment 14.—Take a light, stiff rod, marked off in centimetres (or inches will do). Attach rubber bands, say 4 or 5, to various points of it, and lay it down on a table or board. Draw the bands out, some being on one side of the rod and some on the other, and fasten them so that they all pull along parallel lines. Measure the stretched lengths of the bands.

Now remove the bands and find the weights necessary to stretch them to the same lengths. These give the forces that acted on the rod.

Show that the sum of the pulls on one side is equal to the sum on the other side, and that the algebraical sum of the moments of the pulls about some point in the rod is zero.

Experiment 15.—Take an old drawing-board, or a piece of wood of similar size, about 2 feet by 18 inches. Drive five or six nails into it, and tie strings to them. Attach spring-balances to the other ends of the strings. Lay the wood on a table and pull out the balances in various directions, so that the strings all pull the wood horizontally, with forces indicated by the spring-balances. Fasten the balances so that they continue to exert their pulls steadily.

The wood will, on account of its weight and the friction between it and the table, not take a very definite position of rest, but by moving it about, and noticing the extreme limits of position which it can occupy, the best position can be found.

Now mark the lines along which the strings pull. Take any point in the wood, and carefully measure the perpendicular distances from this point to the strings. Thus determine the moments of the forces acting in the strings about the point.

Show that the algebraical sum of these moments is approximately zero.

Note.—In this, as in many other such experiments, an exact result is not to be expected because of the small errors incidental to the observations. The wood can only be roughly placed in the position which it would occupy if the table were perfectly smooth. The indications of the spring balances cannot be read with absolute accuracy even if we were sure that the balances were absolutely correct. The perpendicular distances cannot be measured with absolute accuracy. Thus, we must be content with finding for the algebraical sum of the moments a very small amount. But the more carefully and accurately the experiment is performed the more nearly will the result be zero, as the theory indicates that it should be.

EXAMPLE.—A light rod 16 inches long rests horizontally with its ends on two supports. It carries masses of 4, 5, and 6 lbs. at points 4, 6, and 8 inches from one end. Find the pressures on the two supports.

[The *downward* pressures *on* the supports will be equal to the *upward* pressures on the rod *from* the supports which, with the weight of the bodies carried by the rod, hold it in equilibrium. In any case of this sort the pressures required will be found by considering the magnitudes of the upward forces due to the supports.

Further, notice in this question that when we have to find two unknown forces which, with others, keep a body in equilibrium, everything being known but the *magnitudes* of these two forces, to find either of them we take moments about a point through which the other acts. We thus get an equation involving only the magnitudes of one force.

The formal solution of the question follows.]

Let **P** and **Q** pounds' wt. be the required pressures. Then the bar is held in equilibrium by the action of the weights of the 4, 5, and 6 pounds acting downwards, and **P** and **Q** pounds' wt. acting upwards, as shown in the figure.

Taking moments about the left-hand end,

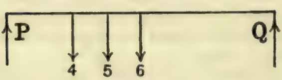

FIG. 64.

$$\mathbf{Q} \cdot 16 = 4 \cdot 4 + 5 \cdot 6 + 6 \cdot 8$$
$$= 94.$$
$$\mathbf{Q} = 5\tfrac{7}{8}.$$
But $$\mathbf{P} + \mathbf{Q} = 15;$$
$$\therefore\ \mathbf{P} = 9\tfrac{1}{8}.$$

Thus the required pressures are **$9\frac{1}{8}$ and $5\frac{7}{8}$ lbs.' wt.**

[To check, we may determine **P** in the same way as **Q** was found, that is, by moments about the right-hand end.

Thus $$\mathbf{P} \cdot 16 = 4 \cdot 12 + 5 \cdot 10 + 6 \cdot 8$$
$$= 146.$$
$$\mathbf{P} = 9\tfrac{1}{8}, \text{ the same as before.]}$$

EXAMPLE.—AB is a light rod 12 inches long. C and D are points on it, such that $AC=3$ inches, $CD=3$ inches. Weights of 3, 4, 6, and 2 pounds are placed at A, C, D, and B. Find at what point the rod will balance.

[We have here to find the resultant of given forces. We are only asked for the *position* of the resultant, that is, the point at which an upward pressure must act on the rod; but the question will be most easily solved by finding the resultant completely, that is, by beginning with its magnitude.

The same is true for many questions of the same sort. Although we may only be asked for the position of a force, as, for instance, that supplied by a support, this is often most simply determined by considering the force fully.]

The upward pressure due to the point of support is 15 lbs.' wt.

Suppose this point to be at x inches from the end A.

Then, by moments about A,

$15x=4 \cdot 3+6 \cdot 6+2 \cdot 12=72.$

$\therefore \; x=4\frac{4}{5}.$

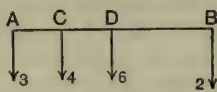

FIG. 65.

The rod will balance **at a point $4\frac{4}{5}$ inches from A.**

We shall now show how the question may be solved without considering the force due to the balancing point. In this we must have an equation not involving this force. To get such an equation we must take moments about the balancing point.

Let the balancing point be x inches from A.

Since the moments of 6 and 2 about C are greater than the moment of the 3 about C, and the moments of the 3 and 4 about D are greater than the moment of the 2 about D, the balancing point must be between C and D. Thus x is between 3 and 6.

By moments about the balancing point,

$$3x+4(x-3)=6(6-x)+2(12-x),$$
$$(3+4+6+2)x=4 \cdot 3+6 \cdot 6+12 \cdot 2,$$
$$15x=72,$$
$$x=4\tfrac{4}{5}.$$

Again we may proceed in a slightly different way, without troubling to consider between what two weights the balancing point is, to begin with. Taking x inches as the distance of this point from A, we write down the *algebraical* sum of the moments of all the forces about the balancing point, and equate them to zero. The moment of the 3 is, of course, $3x$. If x is greater than 3, the moment of the 4 is positive (counter-clockwise) and equal to $4(x-3)$; if x is less than 3, the moment of the 4 is negative and numerically equal to $4(3-x)$. In any case the algebraical value of this moment is $4(x-3)$. Similarly for all the others. The solution would then be as follows:

By moments about the balancing point,

$$3x+4(x-3)+6(x-6)+2(x-12)=0.$$

$$15x=72.$$

$$x=4\tfrac{4}{5}.$$

This method involves troublesome algebraical considerations, but it should be studied, because it illustrates the proposition that the algebraical sum of the moments about the fixed point is zero.

The first method given of solving the problem is probably the one which will be found most useful by the majority of students.

EXAMPLE.—Replace a force of 10 units by two like parallel forces, in the ratio 3 : 4, acting at 6 inches apart.

The sum of the required forces is 10 units, and their ratio 3 : 4, hence they must be $\frac{3}{7}$ of 10 and $\frac{4}{7}$ of 10 units.

The given force must act between the required forces.

Suppose it acts at a distance x inches from the force $\frac{30}{7}$.

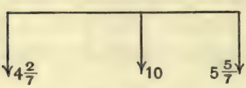

FIG. 66.

Then by moments about a point on this force,

$$10x=\frac{40}{7}\times 6.$$

$$x=\frac{24}{7}=3\tfrac{3}{7}.$$

$$6-x=2\tfrac{4}{7}.$$

The required forces are $4\frac{2}{7}$ **and** $5\frac{5}{7}$ **units acting on opposite sides of the given force parallel to it and in the same sense as it, and at distances from it equal to** $3\frac{3}{7}$ **and** $2\frac{4}{7}$ **inches respectively.**

EXAMPLE.—AB is a light rod fixed by a smooth pin at the end A, and held in a horizontal position by a string BC fastened to B and to a point C vertically above A, and inclined at 30° to the rod. A mass of 1 cwt. is hung on the rod at D, where $AD=\frac{1}{3}AB$. Find the tension in the string.

Let **T** lbs.' wt. be the tension in the string.
Draw AE perpendicular to BC.
By moments about A,

$$\mathbf{T}\,.\,AE=112\,.\,AD\,;$$

i.e.
$$\mathbf{T}\,.\,\frac{AB}{2}=112\,.\,\frac{AB}{3}.$$

$$\mathbf{T}=\frac{224}{3}$$
$$=74\tfrac{2}{3}.$$

The required tension is **74⅔ lbs.' wt.**

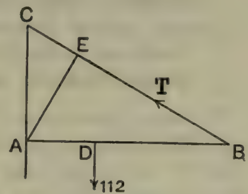

FIG. 67.

EXAMPLE.—A light rod is supported horizontally by resting on a peg at A and under another at B. The pegs are 4 inches apart, and each can just support a pressure of 20 lbs.' wt. without breaking. A weight of 12 lbs. is hung on the rod. Find how far it can be slid along the rod from A before either peg gives way.

For equilibrium of the rod the weight must be on the side of A away from B.

When the weight is x inches from A, suppose the pressures on the pegs A and B are **P** and **Q** lbs.' wt., so that A exerts a force of **P** lbs.' wt. upwards, and B a force of **Q** lbs.' wt. downwards on the rod.

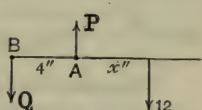

FIG. 68.

Then $$\mathbf{P}=\mathbf{Q}+12.$$

Thus the pressure on A is greater than that on B, and it is A that will be made to give way by moving the weight along.

By moments about B,

$$4\mathbf{P}=(4+x)\,12.$$

$$\mathbf{P}=12+3x.$$

When $\mathbf{P}=20$, $x=2\frac{2}{3}$,

and, since $\mathbf{P}=12+3x$, as long as x is $< 2\frac{2}{3}$, so that $3x$ is < 8, $\mathbf{P}$ is < 20.

Thus A will be on the point of giving way **when the weight is moved as far as $2\frac{2}{3}$ inches from A.**

Example.—$ABCD$ is a rectangle that can turn freely about A. $AD=7$ inches, $AB=5$ inches. Forces of 5 and 10 units act from D to C and from B to D. Find the force along CB which will preserve equilibrium.

[In many questions of this sort, involving geometrical figures, the difficulties are rather those of geometry or mensuration than of mechanics, the mechanical principles involved being very simple. Here, for instance, we have to find the value of the perpendicular from A to BD in order to find the moment of the force in BD about A. The following solution will show how this is done.]

Let the perpendicular from A to BD be x inches.

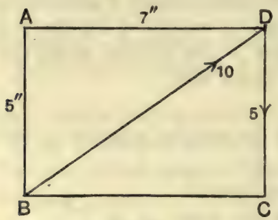

Fig. 69.

Then $x\,.\,BD=7\,.\,5$,

i.e. $x\sqrt{7^2+5^2}=35$,

$$x=\frac{35}{\sqrt{74}}.$$

Moment of 10 about $A=\dfrac{350}{\sqrt{74}}$.

Moment of 5 about $A\;=35$.

Of these the former is the greater, $\because$ 10 is $>\sqrt{74}$.

Thus, required force must act from C to B.

Let it be $\mathbf{P}$ units.

By moments about A,

$$\mathbf{P}\,.\,5=\frac{350}{\sqrt{74}}-35=35\left(\frac{10}{\sqrt{74}}-1\right).$$

$$\mathbf{P}=7\left(\frac{5\sqrt{74}}{37}-1\right).$$

Centre of Parallel Forces.—However the resultant is found, it is seen that the position of the point C does not depend on the common direction of the given forces, that is, on their inclination to AB.

Hence, however **P** and **Q** may be changed in direction, so long as they remain parallel to each other and are always like, or always unlike, and always act at the given points A and B, their resultant will pass through a fixed point C on AB.

C is called the **centre** of the parallel forces **P** and **Q**.

This result may be extended to the case of any number of parallel forces acting at fixed points.

Take, for example, the case of three parallel forces **P**, **Q**, **S**, as shown in the figure.

The resultant of **P** and **Q** is **R**′ at C'.

The resultant of **R**′ and **S** is **R** at C.

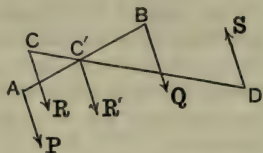

FIG. 70.

Now, whatever be the positions of the lines of **P**, **Q**, **S** with reference to the figure ABD, C' is found by the relation

$$AC' : C'B = \mathbf{Q} : \mathbf{P};$$

and C is found by the relation

$$C'C : CD = \mathbf{S} : \mathbf{R}'.$$

Hence C is a fixed point, however the common direction of the given forces with reference to the figure ABD may be changed.

Obviously the same reasoning may be extended to any number of parallel forces acting at fixed points.

The fixed point for the resultant is always called the **centre** of the given parallel forces.

Couple.—Let **P**, **P** be two equal unlike parallel forces.

These have no single resultant, for they have no translative effect in any direction.

They do, however, have some effect, for they tend to rotate the body on which they act in their plane.

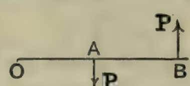

FIG. 71.

Take any point O in the plane of the forces, and draw OAB perpendicular to their lines of action.

Then the algebraical sum of the moments of the forces about O is

$$\mathbf{P} \cdot OB - \mathbf{P} \cdot OA = \mathbf{P} \cdot AB.$$

These two forces are called **a couple.**

The product of either force and the perpendicular distance between them is called **the moment of the couple.**

The perpendicular distance between the forces is called the *arm* of the couple.

The moment of a couple is considered positive or negative according as the couple tends to produce rotation in the positive or negative sense.

We see that the sum of the moments of the forces about any point in their plane is the same, and is equal to the moment of the couple.

Two couples in the same sense and having equal moments will then have the same statical moment, or rotative tendency, about any point in their plane; and neither of them has any translative tendency. Hence, we may infer that these couples have precisely the same effect, and that one may be replaced by the other.

This may, however, be more exactly proved in the following way. We begin by showing that

Two couples acting in a plane, having equal moments in opposite senses, are in equilibrium.

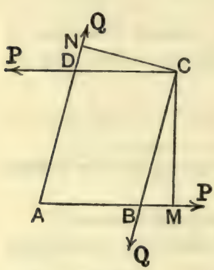

FIG. 72.

Let **P**, **P** be the forces of one couple, and **Q**, **Q** those of the other.

(i) Let the lines of **P**, **P** be inclined to those of **Q**, **Q**.

Let the lines of action of the forces form the parallelogram $ABCD$.

From C draw CM, CN perpendicular to AB, AD.

Then, since the moments of the couples are equal,

$$\therefore \mathbf{P} \cdot CM = \mathbf{Q} \cdot CN.$$

$\therefore$ the resultant of **P** and **Q** along AB, AD passes through C. But it also passes through A.

$\therefore$ the resultant of **P** and **Q** along AB, AD is along AC.

Similarly the resultant of **P** and **Q** along CD, CB is along CA.

And these resultants are equal in magnitude, for the components of one are equal to those of the other and contain the same angle.

$\therefore$ the four given forces are in equilibrium.

(ii) Let the lines of **P**, **P** be parallel to those of **Q**, **Q**.

Draw *ABCD* at right angles to the lines of the forces.

Take the point *O* in *BC*, so that

$$\mathbf{P} . BO = \mathbf{Q} . CO.$$

Then
$$\mathbf{P} . AO = \mathbf{P} . AB + \mathbf{P} . BO$$
$$= \mathbf{Q} . CD + \mathbf{Q} . CO$$
$$= \mathbf{Q} . DO.$$

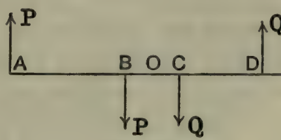

FIG. 73.

Thus, the resultant of **P** and **Q** at *B* and *C* is **P**+**Q** at *O*, and that of **P** at *A* and **Q** at *D* is **P**+**Q** at *O*, these two resultants balancing each other.

Hence, in any case the four forces forming the two couples are in equilibrium.

Since the couple **P**, **P** balances the couple **Q**, **Q**, it is equivalent to any other couple which will do the same; that is, it is equivalent to any couple which has the same moment and is in the same sense as it.

Any number of couples acting in a plane may be replaced by a single couple whose moment is equal to the algebraical sum of their moments.

We know that we may replace a couple by any other having any forces in any direction and positions, provided the moments of the couples are the same.

Now replace all the given couples by others, all having forces equal to **P**; and take the arms of these couples in order along the straight line *AO*, these being measured to the right if the moments are positive, and to the left if they are negative. Thus let the arms be as in the figure, *AB*, *BC*, *CD*, *DE*.

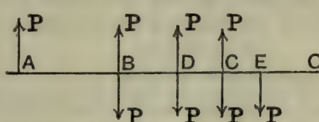

FIG. 74.

Then the couples are **P**, **P** at *A*, *B*, **P**, **P** at *B*, *C*, **P**, **P** at *C*, *D*, **P**, **P** at *D*, *E*.

These are equivalent to **P**, **P** at A, E, that is, to a couple of moment $P.AE$, or $\mathbf{P}(AB+BC-CD+DE)$,

or $$\mathbf{P}.AB+\mathbf{P}.BC-\mathbf{P}.CD+\mathbf{P}.DE,$$

which is the algebraical sum of the given moments.

And this couple may be replaced by any other having the same moment.

Hence the given couples may be replaced by any couple having a moment equal to the algebraical sum of their moments.

EXAMPLE.—Three forces, each equal to **P**, act along the sides AB, BC, CD of a square $ABCD$. Find the magnitude and position of their resultant.

The square may be taken as the polygon for the three forces acting along AB, BC, CD, these forces being represented in magnitude and direction by the sides along which they act.

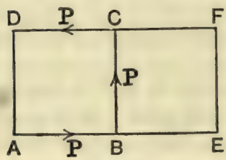

FIG. 75.

Hence their resultant is represented by AD.

Or the resultant is a force **P** parallel to AD.

Now the given forces clearly have a positive moment about A. [Or, together they tend to produce counter-clockwise rotation about A.]

∴ the resultant must meet AB produced.

Let it meet AB at a distance x from A.

Take moments about A.

$$\therefore\ \mathbf{P}.x=\mathbf{P}.AB+\mathbf{P}.AD.$$

$$x=2AB.$$

The resultant is a force **P** along EF, where EF is parallel to AD, and at a distance from it equal to twice the side of the square.

Alternative solution.

Replace the couple consisting of **P** along AB and **P** along CD by a couple consisting of **P** along EF and **P** along CB, where EF is parallel to BC, and $BE=AB$.

This can be done because this couple has the same moment in magnitude and sense as the one it replaces.

Hence the given forces are equivalent to **P** along *BC*, **P** along *CB*, and **P** along *EF*.

But **P** along *BC* and **P** along *CB* neutralize each other.

The given forces are thus equivalent to **P** along *EF*.

Experiment 16.—Take the divided bar and support it by spring-balances near its ends. Note carefully the indications. These are the forces which support the weight of the bar.

Hang on weights, 4 lbs. 5 inches to the right of the left-hand balance, and 5 lbs. 4 inches to the left of the right-hand balance. The additional forces shown by the balances and the weights are in equilibrium. It will be found that these additional indications are 4 and 5 lbs.' wt. Thus the bar is acted on by the two couples, one of forces 4 lbs.' wt. at 5 inches apart, the other of forces 5 lbs.' wt. at 4 inches apart. The moments of these couples are equal, and they act in opposite senses.

Other weights in other positions may be tried which will give couples; for instance, 2 lbs. 6 inches to right of one balance and 3 lbs. 4 inches to left of the other, and so on.

Exercises IV.

1. A two-foot rod has a 4 and a 7 lb. wt. attached to its ends. On what point will it balance?

2. A crow-bar 6 feet long is used to raise a heavy stone, one end resting on the ground and pressing the stone upwards at a point $1\frac{1}{2}$ inches from this end. If the hands supply an upward pressure of 80 lbs.' wt. at the other end, what is the pressure on the stone, the weight of the bar being neglected?

3. Two men carry a weight of 200 lbs. on a pole 8 feet long, the weight being at first in the middle. If the weight slips 6 inches nearer to one man, find the additional load he has to carry.

4. In the same question show that wherever the weight is, to begin with, if it slips 6 inches nearer to one man, he has in consequence the same additional load to carry.

5. *ABCD* is a square, 10 inches in the side; it can rotate freely about *A*. Forces 3, 6, 9 oz. wt. act along *AB*, *BC*, *CD*. A long straight arm is attached in line with *AD*. Find where on this arm, and at right angles to it, a force of 1 oz. wt. must act to keep the square fixed, and what the consequent pressure on the fixed point *A* is.

6. The distance between the axles of a bicycle is 3 feet 4 inches, and the weight of a rider weighing 11 stone rests at a point 1 foot 2 inches in front of the back wheel axle. What portion of his weight is carried by each wheel?

7. The distance, measured horizontally, from the axle of a 2-wheel cart to the saddle tugs is 11 feet. The load, 3 cwt., falls 8 inches in front of the axle. What is the pressure on the horse's back?

8. Two strings, each of which can just carry a weight of 11 lbs., support a rod AB 12 inches long in a horizontal position by its ends, the strings being vertical. A weight of 10 lbs. is hung at a point 4 inches from A. What is the least distance from A that a weight of 7 lbs. can be placed without breaking the string at A?

9. A man carries a bundle weighing 25 lbs. by a light stick over his shoulder. He holds the stick at 10 inches from his shoulder and 2 feet from the point at which the bundle is attached. What is the pressure on his shoulder, and the force which he must exert downwards with his hand?

10. $ABCD$ is a square pivoted at A. Forces of 5 and 7 lbs.' wt. act along BC and DB. Find what force must act at right angles to AD through its middle point to keep the square at rest.

11. ABC is a triangle in which $AB=AC=13$, and $BC=10$. A force of 26 units acts along CA. What force must act from A to the middle point of BC to maintain equilibrium if the point B is fixed?

12. A rod ACB balances at C. A weight W at A is balanced by W_1 at B, and W at B is balanced by W_2 at A. Show that

$$W^2 = W_1 W_2.$$

13. A rod balances at O. A weight W placed at distance a from O is balanced by W' at distance b from O. If W is placed at distance b from O, show that, to balance it, W' must be placed at distance $\frac{b^2}{a}$.

14. Draw $ABCDEF$ to represent a light rod. $AB=4''$, $BC=6''$, $CD=2''$, $DE=4''$, $EF=7''$. The rod is supported by two vertical strings of the same sort at B and E, and has weights of 3, 8, and 1 lbs. at A, D, and F. A weight of 6 lbs. at C just causes one string to break. Which string is it? and what weight is just sufficient to break either string?

15. A beam 10 feet long is carried on two supports at its ends, and is loaded in any manner. If the support at one end is moved two feet towards the middle of the beam, show that the pressure it sustains becomes greater by one-fourth than what it was at first.

16. A body, capable of motion about a fixed axis, is acted upon by two forces equal to the weights of 10 lbs. and 8 lbs. respectively. Show how these forces may be made to act so that the body is in equilibrium. Illustrate your answer by a diagram. (Oxford Jr. Local, 1896.)

17. A cyclist weighing 150 lbs. puts all his weight on one pedal of a bicycle when the crank is horizontal, and the bicycle is kept from moving forwards. If the length of the crank is 6 inches, and the radius of the chain-wheel is 4 inches, find the tension of the chain. (Camb. Jr. Loc., Stat. Dyn. and Hydro., 1898.)

SUMMARY.

The resultant of two parallel forces can be fully determined by theory, starting from the parallelogram of forces.

The necessary and sufficient conditions for equilibrium of a body free to turn about a fixed point, and acted on by any forces, can be found by theory, making use of

(1) the parallelogram of forces,

(2) the principle that two forces are in equilibrium (or balance each other) when, and when only, they are equal and opposite.

Centre of Parallel Forces.—If any number of parallel forces act through fixed points, and are turned about in any manner, always, however, remaining parallel to each other and acting through the same fixed points, their resultant passes through a fixed point. This is called the **centre of the parallel forces.**

A couple consists of two equal parallel forces acting in opposite senses.

A couple has no single resultant.

The **moment of a couple** means the product of one of the forces and the perpendicular distance between the two forces.

Two couples in the same plane have the same statical effects if they act the same way, and their moments are equal.

Any number of couples acting in a plane may be replaced by a single couple whose moment is equal to the algebraical sum of the moments of the given couples.

CHAPTER V.

CENTRE OF GRAVITY. MASS. DENSITY. SPECIFIC GRAVITY.

Centre of Gravity or Centre of Mass.—Imagine two indefinitely small heavy particles placed at the points A and B, and rigidly connected together by a perfectly light connexion, so that their distance, AB, remains unalterable. Let the weights of the particles be $\mathbf{W}_1$ and $\mathbf{W}_2$.

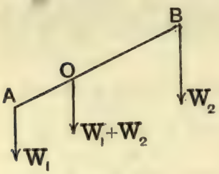

FIG. 76.

Then, however the straight line AB may be inclined to the vertical, the resultant of the forces $\mathbf{W}_1$ and $\mathbf{W}_2$ is a force parallel to them, that is, vertical, and acting through a fixed point O in AB, so that

$$\frac{AO}{OB}=\frac{\mathbf{W}_2}{\mathbf{W}_1},$$

that is, so that O is the centre of parallel forces $\mathbf{W}_1$, $\mathbf{W}_2$ at A and B.

Now, if the arrangement or system of two particles be supported in any manner, the forces which support it, since they produce equilibrium with $\mathbf{W}_1$ and $\mathbf{W}_2$, would produce equilibrium with their resultant $\mathbf{W}_1+\mathbf{W}_2$. So that exactly the same forces would be required to support the system as if the whole weight were collected at O.

In the same manner we may suppose that there is also a third particle of weight $\mathbf{W}_3$ at C, the three being rigidly connected together. Let O be now the centre of parallel forces $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ at A, B, C. Then the arrangement of three particles has weights whose resultant is always the force $\mathbf{W}_1+\mathbf{W}_2+\mathbf{W}_3$

acting vertically downwards through *O*. Hence the arrangement will always require just the same forces to support it as if all the particles were collected at *O*.

It is clear that we may extend this reasoning to any number whatever of heavy particles. Suppose such particles to be rigidly connected together at points *A*, *B*, *C*, *D*, etc., so that however the system may be turned about, the relative configuration remains unaltered; and suppose their weights are $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, etc. Let *O* be the centre of parallel forces $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, etc., at *A*, *B*, *C*, *D*, etc. Then *O* is a point fixed relatively to the system. And the system will require the same forces to support it, and will produce the same pressures on any given external supports as if the whole weight acted through *O*.

The point *O* found in this way is called the *Centre of Gravity*, or *Centre of Mass*, of the given system of particles.

Centre of Gravity of three equal heavy Particles.—To find the centre of gravity of three equal heavy particles at the angular points of a triangle *ABC*.

Let **W** be the weight of each particle.

The weights **W**, **W** at *B* and *C* are equivalent to a weight 2**W** acting at *D*, the middle point of *BC*.

Join *AD*. And take the point *G* in *AD* so that $DG = \frac{1}{3} . DA$.

Then the weights 2**W** at *D* and **W** at *A* are equivalent to 3**W** acting at *G*.

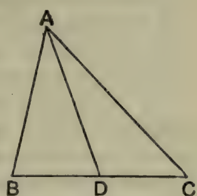

FIG. 77.

Hence *G* is the required Centre of Gravity.

Centre of Gravity of a Solid Body.—The same reasoning as above may be extended to a solid body. The solid body may be supposed to be made up of an indefinitely large number of indefinitely small particles. The weight of each of these particles acts through the point at which the particle is situated. So we have a number of parallel forces acting through fixed points. Their resultant is equal to the sum of the weights, and acts through the centre of the parallel forces. This point is the Centre of Gravity of the body. The body will then always produce the same statical effect, or will act with the same forces on all external bodies, supports, etc., as if its whole weight really acted through its Centre of Gravity.

It should be noticed that *to the body itself* it makes a difference whether the weight acts all through the Centre of Gravity or not, for the forces holding the parts of the body together will be different in the two cases. Consider, for instance, a long uniform heavy beam carried on props at its ends. The Centre of Gravity of the beam is at its middle point. The pressures on the props will be the same as if the whole weight of the beam acted at its middle point. But the beam itself would be differently affected. For the weight would have a much greater tendency to break the beam if it all acted at its middle point than if it were uniformly distributed along the beam as is actually the case.

Definition of Centre of Gravity.—We may define Centre of Gravity in the general case as follows:

The Centre of Gravity, or Centre of Mass, of a body or system of particles is the point through which the statical resultant of the weight of all the particles of which the body or system is composed always acts in all positions of the body.

A very important property of the Centre of Gravity is that a body will balance in any position in which it may be placed on its Centre of Gravity, this point being fixed.

For the weights of all the particles of which the body is composed give rise to a resultant passing through the Centre of Gravity, and this can have no tendency to turn the body any way about the Centre of Gravity. The body therefore remains in equilibrium.

If a system of heavy particles is arranged along a straight line, the Centre of Gravity of the system is in this line.

For the centre of a system of parallel forces at points in a straight line is obviously in the same line.

We may find the Centre of Gravity of such a system by any method for finding the point on which the system will balance.

EXAMPLE.—AB is a light rod. Weights of 3, 5, 7, 9 lbs. are placed at A, C, D, B; where $AC = CD = DB = 3$ inches. Find the Centre of Gravity of the system.

Let the Centre of Gravity be at x inches from A. The resultant of the weights of the masses is a force equal to the sum of the weights acting through the Centre of Gravity.

Then by moments about A,

$$x(3+5+7+9)=3.5+6.7+9.9,$$

$$24x=138,$$

$$x=5\tfrac{3}{4}.$$

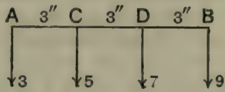

FIG. 78.

The Centre of Gravity is a point on the rod $5\frac{3}{4}$ inches from the end A.

The term Centre of Gravity is frequently abbreviated into C.G.

Centre of Gravity of a straight uniform Rod.—A uniform rod in Mechanics means one in which the weight per unit of length is the same throughout the length of the rod; that is, such that if the rod be divided up into any number of equal lengths, however short, the weight of each of these lengths is the same.

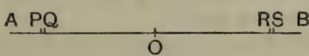

FIG. 79.

Let AB be the rod and O its middle point.

Imagine the rod to be divided into an indefinitely large number of elementary pieces of equal length.

Let PQ, RS be two of these elements equally distant from O. Now each of these pieces being of infinitesimal extent may be supposed to have its weight all acting at one point, *the point* at which the piece is situated.

The weights of the two pieces have for resultant a force equal to the sum of the weights acting at O.

In the same way all the elementary pieces may be grouped together in pairs like these two, the resultant of the weight of each pair acting through O.

Thus the weight of the whole rod acts through O, or O is the required Centre of Gravity.

EXPERIMENT 17.—Take a heavy rod and find its weight, and find on what point it will balance. This point is its C.G., and is, of course, the middle point of the rod if it is uniform. Mark the C.G.

Suspend the rod horizontally by two spring balances on opposite sides of the C.G., and at different distances from it. Note the indications of the balances.

Show that the forces indicated have for equilibrant a force equal to the weight of the rod acting through its C.G.

Suppose, for example, the rod weighs 8 lbs., and the balances are attached at points 6 inches and 18 inches from the C.G.

The indications in the balances should then be 6 and 2 lbs.' wt.

In many questions in Statics about rods no account is taken of any dimensions except the length, that is, the thickness in any direction is supposed to be indefinitely small as compared with the length, or the rod is considered as a *material straight line.*

Any straight uniform rod will have its C.G. in its middle section, that is, in the section of it at right angles to its length midway between its ends. This may easily be seen by reasoning similar to the above. In a question in which we take no account of the extent of this section, but consider it all as one point, the C.G. of the rod is at this point.

A heavy bar or rod not otherwise described is to be supposed uniform.

EXAMPLE.—A uniform beam 10 feet long weighing 50 lbs. rests horizontally on two supports at its ends, and carries a weight of 30 lbs. at a point 4 feet from one end. What are the pressures on the supports?

Let AB represent the beam, C its middle point.

Let **P**, **Q** lbs.' wt. be the pressures on the supports at A and B.

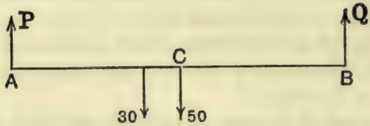

FIG. 80

The weight of the beam may be supposed to act at C.

By moments about B,

$$\mathbf{P} \cdot 10 = 50 \cdot 5 + 30 \cdot 6,$$

$$\mathbf{P} = 43.$$

And $\mathbf{P}+\mathbf{Q}=80.$

$\therefore \ \mathbf{Q}=37.$

The required pressures are **43 and 37 lbs.' wt.**

EXAMPLE.—*AB* is a beam. If weights 5, 10, 5 lbs. are attached at points 3, 8, 9 feet from *A*, the beam will balance on a point 6 feet from *A*. If the 10 lb. weight is removed, the beam will balance on a point 5 feet from *A*. What is the weight, and where is the C.G. of the beam?

Let the weight be **W** lbs.' wt., and the distance of the C.G. from *A* x feet.

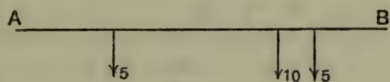

FIG. 81.

In the first case, the upward pressure at the balancing point is $\mathbf{W}+20$ lbs.' wt.

Then by moments about *A*,

$$\mathbf{W}x+5\,.\,3+10\,.\,8+5\,.\,9=(\mathbf{W}+20)\,6.$$

In the second case, the upward pressure at the balancing point is $\mathbf{W}+10$ lbs.' wt.

By moments about *A*,

$$\mathbf{W}x+5\,.\,3+5\,.\,9=(\mathbf{W}+10)\,5.$$

These equations reduce to

$$\mathbf{W}x+140=6\mathbf{W}+120,$$

$$\mathbf{W}x+60=5\mathbf{W}+50,$$

$$\mathbf{W}=10,$$

$$x=4.$$

The beam **weighs 10 lbs., and its C.G. is 4 feet from the end *A*.**

Combined Centre of Gravity of several Bodies.—If we know the weights and C.G.s of several bodies, the combined C.G. of them when rigidly joined together in any manner, and considered as one body, can easily be found.

Suppose several bodies of weights $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, etc., have C.G.s at A, B, C, etc.

The C.G. of the combination is the point at which the resultant of the weights of all the separate elements acts.

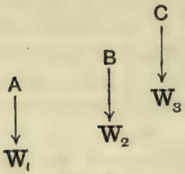

Fig. 82.

Now the weights of the elements of the first body have resultant $\mathbf{W}_1$ at A, the weights of the elements of the second body have resultant $\mathbf{W}_2$ at B, and so on.

The final resultant is that of these resultants.

Thus, to find the point at which the entire weight may be supposed to act, we may suppose that the bodies are small particles, of weights $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, etc., situated at A, B, C, etc.

The final Centre of Gravity is the same as that of particles, of weights equal to those of the bodies, and situated at the Centres of Gravity.

Example.—Find the C.G. of a body composed of three solid spheres rigidly joined with their surfaces in contact, and their centres in a straight line, the radii of the spheres being 3, 4, 5 inches.

Let A, B, C be the centres of the spheres.

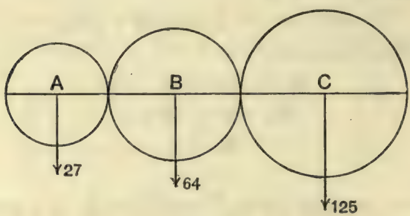

Fig. 83.

The volumes, and $\therefore$ the weights of the spheres are proportional to the cubes of their radii, and may be taken as 27, 64, 125.

The C.G. lies on the line ABC. Let it be x inches from A.

By moments about A,

$$x(27+64+125)=7 \,.\, 64+16 \,.\, 125,$$

$$x=\frac{7 \,.\, 64+16 \,.\, 125}{216}=\frac{7 \,.\, 8+2 \,.\, 125}{27}$$

$$=\frac{306}{27}=\frac{34}{3}=11\tfrac{1}{3}.$$

The C.G. is **between** A **and** C, **at** $11\frac{1}{3}$ **inches from** A.

[This may be checked by finding the distance, y inches, of the C.G. from C.

Thus $$216y=64 \,.\, 9+27 \,.\, 16,$$

$$y=4\tfrac{2}{3},$$

a result which agrees with the other.]

EXAMPLE.—A uniform rod AB of weight **W**, 8 feet long, can turn freely about A. It is held in a horizontal position by means of a string attached to B, and making an angle of 45° with the rod. A body of weight **W** is attached to the rod at 6 feet from A. Find the tension in the string and the reaction at A.

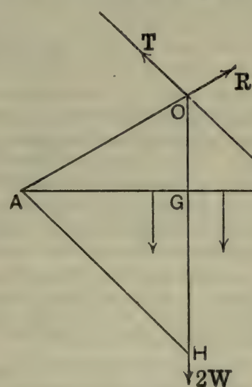

FIG 84.

The resultant of the weights is a force 2**W** acting vertically downwards through G, a point 6 feet from A.

Let **T** be the tension in the string, and **R** the reaction at A.

Let the vertical through G meet the string in O.

Then the third force **R**, keeping the rod in equilibrium with **T** and 2**W**, must also act through O.

Draw, through A, AH parallel to OB, meeting the vertical through G in H. Then HAO is triangle of forces.

$$\therefore \frac{\mathbf{T}}{HA}=\frac{\mathbf{R}}{AO}=\frac{2\mathbf{W}}{OH}.$$

Now $AG=GH=5$,

and $GO=GB=3$.

$$\therefore\ HA=\sqrt{50},\text{ and } AO=\sqrt{34}.$$

$$\therefore\ \frac{\mathbf{T}}{\sqrt{50}}=\frac{\mathbf{R}}{\sqrt{34}}=\frac{\mathbf{W}}{4}.$$

The required tension and reaction are

$$\frac{\mathbf{W}\sqrt{50}}{4}\text{ and }\frac{\mathbf{W}\sqrt{34}}{4}.$$

In this question it should be noticed how the fact is used that the three forces keeping a body in equilibrium must all pass through one point if they are not all parallel. By noticing this we are able to get the direction of **R**, and thus get a triangle of forces.

The question might have been otherwise solved by the following method.

Find **T** by taking moments about A.

Then find **R** as the equilibrant of 2**W** and **T** inclined at 135°.

Exercises V. a.

1. Find the C.G. of weights 4, 5, 7, 2, 11 ounces placed at distances of 1 inch apart along a straight rod.

2. A bar 1 foot long, weighing 2 lbs., has a 5 lb. wt. attached to one end. What weight must be attached to the other end to make it balance at a point 2 inches from the former end?

3. A rod is 14 inches long and weighs 2 ounces, and will balance on a point 6 inches from one end. Where must a ½-ounce weight be attached to make it balance at its middle point?

4. A uniform plank 16 feet long, weighing 5 lbs. per foot, rests on a wall with 5 feet projecting beyond the end. How far can a boy, weighing 5 stone, walk along the plank from the end of the wall without upsetting the plank?

5. An iron crowbar 5 feet long has a 7 lb. weight attached to one end, and balances about a point 2 feet from that end. What is the weight of the bar?

6. A rod weighs 50 grams. With a weight of 4 grams hung on one end it balances at a point 1 metre from this end. What is its length?

7. A rod weighs 50 grams. With a weight of 4 grams hung on one end it balances at a point 1 metre from the other end. What is its length?

8. A beam AB weighs 50 lbs., and is supported on two props at its ends. If a weight of 70 lbs. is placed 2 feet from A, the pressure on B is 30 lbs.; and if a weight of 60 lbs. is placed at 2 feet from B, the pressure on B is 70 lbs. What is the length of the beam, and where is its C.G.?

9. ABC is a triangle in which $AB=AC=13$ inches, $BC=10$ inches. What are the distances from the points A, B, C of the C.G. of 3 equal heavy particles placed at A, B, C?

10. ABC is a triangle in which $AB=\sqrt{5}$, $AC=\sqrt{2}$, $BC=3$. Three particles of weights 3, 2, 1 are placed at A, B, C. Show that the C.G. of the three particles is at the middle point of the perpendicular from A on BC.

11. D, E, F are the middle points of the sides of the triangle ABC. Show that the C.G. of three equal particles at D, E, F is the same as that of three equal particles at A, B, C.

12. In the triangle ABC three particles of weights 2, 3, 3 are placed at A, B, C. If $AB=AC=5$, and $BC=6$, find the distances of the C.G. of the particles from A, B, C.

13. Two uniform rods 4 and 3 feet long, and weighing 6 and 8 lbs. respectively, are joined in a straight line. Find their Centre of Gravity.

14. A rod of circular section is of uniform thickness throughout half its length, and of half the thickness throughout the other half of its length. Show that the weight which must be attached to the smaller end to make the rod balance about its middle point is $\frac{3}{10}$ of the weight of the rod.

15. Three circular discs cut from the same plate are fastened by their edges, with their centres in a straight line. Their diametres are 2, 4, and 6 inches. Where is their C.G.?

16. Three circular discs of radii 3, 5, and 7 units, of the same material and of thicknesses proportional to 5, 3, and 1, are connected by their edges with their centres in a straight line. How far from the centre of the smallest disc is the combined Centre of Gravity?

Mass.—The mass of a body is the quantity of matter that it contains.

The mass of a body is properly measured by the difficulty of setting it in motion by the action of a given force, that is, by its *inertia.* This, however, is a question for the science of the action of forces in producing motion.

For questions in Statics we may suppose the mass of a body to be measured by its weight. For, in ordinary circumstances, the weight is always proportional to the mass. A body of mass

1 lb. has a definite weight which we call a pound weight; a body of mass 2 lbs., whether made of the same or of a different material, has just twice as much weight, and so on.

Mass and Weight compared.—Great care must be taken to distinguish between the *mass* and the *weight* of a body. These are quantities which may very easily be confused, because, in general, weight is proportional to mass, and weight is employed to determine mass, and the same number is used to indicate the weight and the mass of a body. Thus we weigh a body, and if we find that it has 5 lbs.' weight, or "weighs 5 lbs.," we infer that its mass is 5 pounds. Then the number 5 denotes the mass in pounds and the weight in lbs.' weight.

Mass and weight are, however, entirely different quantities.

Mass, we may say for short, as has been said above, means quantity of matter, and its true significance belongs to Dynamics, or the science of motion.

Weight, on the other hand, *is a force*, namely, the force with which the earth attracts the body to itself.

Again, the mass of a body is a perfectly unalterable quantity as long as the body remains the same, that is, has nothing added to or taken from it: but the weight of a body may, and does to a slight extent, vary as it is taken from one locality to another, for it depends on the *latitude* in which the body is, and on its *height above sea-level*.

If the body were taken right away from the influence of the earth, it would lose its weight completely. If it were placed on the surface of another heavenly body, what may then be called its weight, the attraction of this body on it, would, as a rule, be very different from its weight on the earth.

Mass and weight, then, are two entirely different sorts of things. Mass is invariable, and is what we may call an *inherent property* of a body; weight is variable, and is an *accidental circumstance*; although within the limits of our experience, the two are practically always in direct proportion to each other; and this fact is employed to measure mass, because weight is easily determined.

Density.—If two bodies are such that one weighs more, bulk for bulk, than the other, or, as we have seen, has a greater mass, bulk for bulk, than the other, it is said to be denser, or to have a greater density, than the other. The exact definition of density is as follows :

The density of a substance is its mass per unit of volume.

The measure of a density will depend upon the units of mass and of volume employed.

For instance, using a foot and a pound as units, the density of water means the number of pounds in a cubic foot, and is about 62·5.

But if we use a centimetre and a gram as units of length and mass, the density of water is the mass in grams of a cubic centimetre, and is, approximately, 1.

It must be noticed that *density* is a quality of a *substance*, without reference to the quantity of it in question. But we can only speak of the *mass* of a definite amount of a given material. Thus, we may speak of the density of lead, but of the mass of a definite quantity of lead.

Specific Gravity.—It is often convenient to compare the density of a substance with that of another substance used as a standard of comparison. Thus, water has a certain density measured by the number of pounds mass in a cubic foot of water. This number is about 62·5. The number of pounds mass in a cubic foot of iron would be found to be about 7 times as great, or about 441. This number measures the density of iron. Thus, the density of iron is 7 times as great as that of water.

Water is used as the standard substance with which other substances are compared. The ratio of the density of the iron to that of water is called the *relative density* of the iron, or sometimes its *specific gravity.*

It is clear that if, instead of using cubic feet of water and iron, we used any other equal volumes of them, the ratio of the mass of iron to the mass of water would always be the same.

Thus, the specific gravity may be defined as the ratio of the masses of any equal volumes.

We have the following general definition :

The specific gravity or relative density of a substance is:

(*a*) *the ratio of its density to that of water*, *or*

(*b*) *the ratio of the mass of any volume of it to that of an equal volume of water.*

The name Specific Gravity is commonly abbreviated by the letters S.G.

In the British system,

$$\text{S.G. of a body} = \frac{\text{density of body}}{\text{density of water}}$$

$$= \frac{\text{density of body}}{62 \cdot 5},$$

because the number of pounds of water in a cubic foot = 62·5.

Or, $\qquad$ Density of body = S.G. of body × 62·5.

In the metric system,

$$\text{S.G. of body} = \frac{\text{density of body}}{\text{density of water}} = \text{density of body},$$

because the number of grams of water in a cubic centimetre = 1.

Exercises V. b.

1. Explain what is meant by the statement that the specific gravity of lead is 11.

Find the volume occupied by 25 lbs. of lead, assuming that 1 cubic foot of water weighs 1000 oz. (Oxford Locals, 1897.)

2. Define density. What is the weight of a brick 25 cm. long, 12 cm. wide, and 8 cm. thick? (Specific gravity of the brick = 2·5, weight of 1 c.c. of water = 1 gram.) (Camb. Jr. Loc., Stat. Dyn. and Hydro., 1897.)

A lamina is a figure consisting of a plane or flat sheet of some material, such as cardboard or metal, so thin that no account need be taken of its thickness.

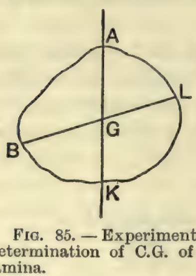

FIG. 85. — Experimental determination of C.G. of a lamina.

Every portion of the material of such a body may be supposed to lie in one plane, the plane of either surface. The Centre of Gravity, therefore, being the centre of parallel forces acting at points of this plane, lies in the plane too.

The Centre of Gravity of a lamina may easily be found by an experiment depending on the principle that, if a body is freely suspended from a point, its Centre of Gravity must be in a vertical line with the point of suspension.

Suspend the body by any point, such as A, and draw on its surface the vertical straight line AK passing through A.

Again, suspend it at B, and draw the vertical straight line through B.

The C.G. is in AK and in BL, and is therefore at G, the point where AK and BL intersect.

For accuracy, the points A and B should be so chosen that AK and BL make a considerable angle with each other. For if the angle between these lines is small, a small error in the position of either of them causes a considerable error in the position of G.

A uniform lamina is one in which the weight per unit area is the same throughout, or such that if it be divided into any number of equal pieces, however small, the weights of all the pieces are equal.

Centre of Gravity of Uniform Parallelogram.—Divide the parallelogram $ABCD$ into indefinitely narrow strips, such as KL, by means of straight lines drawn parallel to AD.

The C.G.s of all these strips are at their middle points, and these points all lie on the straight line joining the middle points of AD and BC.

Thus the C.G. of the parallelogram is on EF.

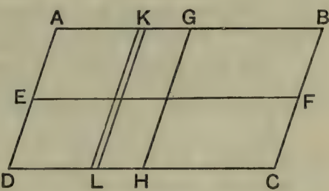

FIG. 86.—C.G. of parallelogram.

Similarly it is on GH, the straight line joining the middle points of AB and DC.

Therefore the Centre of Gravity of the parallelogram is the point of intersection of the straight lines joining the middle points of opposite sides.

This is the same as the point of intersection of the diagonals.

Centre of Gravity of Uniform Triangular Lamina.—Divide the triangle ABC into indefinitely narrow strips, such as KL, by means of straight lines drawn parallel to BC.

The C.G.s of all these strips are at their middle points, and these points all lie on a straight line AD drawn from A to the middle point of BC.

Therefore the C.G. of the triangle lies on AD.

Similarly it lies on the straight line joining B to the middle point of AC.

Hence the C.G. of the triangle is at the point of intersection of any two of the straight lines joining angular points to the middle points of the opposite sides.

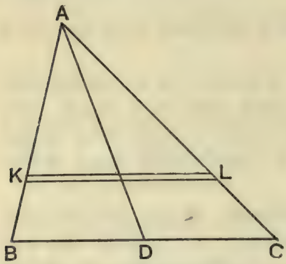

FIG. 87.—C.G. of triangle.

The straight lines joining the angular points of a triangle to the middle points of the opposite sides are called the *medians* of the triangle.

The result obtained above for the position of the C.G. of a triangle affords an indirect proof that the three medians meet at one point, or are *concurrent.*

Another construction is often useful for finding the Centre of Gravity of a triangle.

Draw the medians AD, BE meeting in G. Join DE.

Then, $\because$ D and E are the middle points of CB, CA,

$\therefore$ DE is parallel to BA, and $DE = \frac{1}{2}BA$.

By similar triangles GDE, GAB,

$$\frac{DG}{GA} = \frac{DE}{BA} = \frac{1}{2}.$$

$$\therefore \; DG = \tfrac{1}{2}GA.$$

$$\therefore \; DG = \tfrac{1}{3}DA.$$

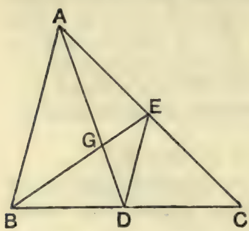

FIG. 88.—C.G. of triangle.

This leads to the following construction for G:

Draw any median, such as AD.

Take DG equal to $\frac{1}{3}DA$. Then G is the C.G. of the triangle.

From either of these constructions for G it is seen that

The Centre of Gravity of a uniform triangle is the same as that of three equal heavy particles at its angular points.

Centre of Gravity of the Perimeter of a Triangle.—Suppose the three sides forming the perimeter of the triangle ABC to be made of the same uniform material, as, for instance, when a uniform piece of wire is bent into the shape of the three sides of a triangle.

Let the lengths of BC, CA, AB be a, b, c.

Make D, E, F the middle points of the sides.

The masses of the sides are proportional to a, b, c, and have their C.G.s at D, E, F.

Let the length of the perpendicular from A on BC be h. Then the length of perpendicular from D on EF is $\frac{h}{2}$.

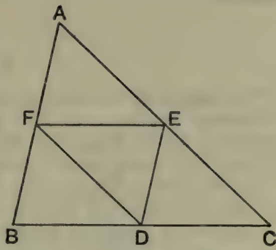

FIG. 89.—C.G. of perimeter of triangle.

If distance of C.G. of perimeter from EF is x, by moments about EF,

$$x \,.\, (a+b+c)=\frac{h}{2} \,.\, a.$$

$$x=\frac{ha}{2(a+b+c)}=\frac{\text{area of triangle}}{a+b+c}.$$

Similarly, the distance of the C.G. of the perimeter from FD and DE are each equal to this same quantity. Thus the C.G. is equidistant from the sides of the triangle DEF.

Or, the required C.G. is the centre of the circle inscribed in the triangle formed by joining the middle points of the sides of the given triangle.

Centroid.—If we take any geometrical figure and suppose it to be a uniform lamina, or a lamina of uniform mass per unit area, and find its Centre of Gravity on this supposition, the point so found is called the Centroid of the figure.

EXPERIMENT 18.—Take a piece of stout cardboard and cut out a figure of irregular shape. Suspend it from a point in the edge, and hang a small weight by means of a string from the same point. Draw, by the help of this string, the vertical through the point of suspension. Again, suspend the cardboard from another point, so that the line first drawn rests about horizontal. Draw the new vertical through the second point of suspension. Thus find the C.G., which is the point of intersection of these two lines.

Show that from whatever point the cardboard is suspended, the C.G. rests vertically below this point.

Show that if a hole is made through the C.G., and the cardboard be carried on a nail passed through this hole, it will rest in any position.

Summary.

The **Centre of Gravity** or **Centre of Mass** of given particles rigidly connected together, or of a body, is the point through which the resultant of the weights of all the infinitesimal particles always acts, however the body may be turned about.

A body will balance on its Centre of Gravity if this point is fixed, in whatever position the body may be placed.

The forces acting on a body and holding it in equilibrium are the same as if the weight of the body acted through its Centre of Gravity.

The C.G. of a straight uniform rod supposed to have no thickness is at its middle point.

The combined C.G. of several bodies is found by supposing each to be concentrated at its own C.G.

The **mass** of a body is the quantity of matter that the body contains. It must be carefully distinguished from **weight**, which is the force with which the earth attracts the body.

Mass is unalterable; weight depends slightly on latitude and altitude.

The **density** of a body is its mass per unit volume. Its measure depends on the unit of mass and on the unit of length employed.

The density of water, using pounds and feet, is about 62·5.

The **specific gravity** of a substance is the ratio of its density to that of water, taking water as the standard substance.

It follows that, for any substance:

In British system *density* = S.G. × 62·5;

In metric system *density* = S.G.

The *Centre of Gravity of a lamina* can be found experimentally by suspending it in turn from two points, and drawing the vertical through each of these points while it is suspended by it, and taking the point of intersection of these verticals.

The *C.G. of a uniform parallelogram* is the intersection of its diagonals.

The *C.G. of a uniform triangle* is the point of intersection of its medians.

CHAPTER VI.

CENTRE OF GRAVITY.—(Continued.)

Centre of Gravity of several Particles in one Plane.—Suppose we have several particles of given weights, $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$, etc., in a plane, and suppose their distances from a given line in the plane to be h_1, h_2, h_3, etc. Then we may find the distance h of the C.G. from the line as follows :

Suppose the plane containing the line and the particles to be placed horizontally, and consider the moments of the weights about the line.

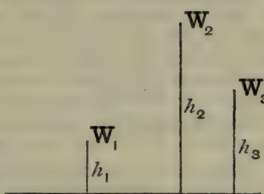

Fig. 90.

The moment of the whole weight $\mathbf{W}_1+\mathbf{W}_2+\mathbf{W}_3+\ldots$ about the line must be the same as the sum of the moments of the separate weights.

Thus

$$(\mathbf{W}_1+\mathbf{W}_2+\mathbf{W}_3+\ldots)h=\mathbf{W}_1h_1+\mathbf{W}_2h_2+\mathbf{W}_3h_3+\ldots.$$

$$\therefore\ h=\frac{\mathbf{W}_1h_1+\mathbf{W}_2h_2+\mathbf{W}_3h_3+\ldots}{\mathbf{W}_1+\mathbf{W}_2+\mathbf{W}_3+\ldots}.$$

It may happen that some of the particles are on one side of the line and some on the other. Then the moments of the weights about the line are in opposite senses. Thus the moments of the weights on one side of the line must be counted positive, and of those on the other side negative, in the sum $\mathbf{W}_1h_1+\ldots$.

We may decide to call all the distances on one side of line positive, and those on the other side negative. Then the sign of *h*, got from the formula given above, will determine on which side of the line the Centre of Gravity is.

If the distances of the particles from another straight line not parallel to the first are also known, we can determine the distance of the C.G. from this straight line as well, and thus completely determine its position.

Suppose the two given straight lines are called Ox, Oy, and are at right angles to each other. It is clear that the position of a point in the plane is fixed if we know its distances from Ox and Oy, and on which side of each of these it lies. We may call distances on one side of each of these lines positive, and those on the other side negative.

Ox, Oy are then called *Axes*, and the distances of any point from them, taken with their proper signs, are called the *Co-ordinates* of the point.

Experiment 19.—Suspend the divided bar at its middle point *O*, so that it balances. Place any weights at known distances to the right of *O*, and balance these by a weight placed at a suitable point on the left of *O*. For instance, if we use weights 1, 2, 3 lbs. at distances 12, 6, and 8 inches from *O* respectively, these could be balanced by 4 at 12 inches to the left of *O*.

Now find the C.G. of all the weights on the right of *O*, and place them all at it. In the case mentioned it is 8 inches from *O*.

Notice that these are still balanced by the same weight on the left in the same position as before.

Experiment 20.—Take a piece of stiff cardboard and suspend it by two strings at points *A* and *B* at its edges, so that *A* and *B* are at the same horizontal level. Draw the straight line *AB*. Place several weights on one side of *AB* at measured distances from it. For instance, we may use 10, 20, 40, 50 grams at distances 16, 20, 7, 12 centimetres from *AB*. And place a weight on the other side of *AB* in a suitable position to make the cardboard balance horizontally.

Now find the distance from *AB* of the C.G. of the first-named weights. In the given case it is 12 centimetres.

Draw a line parallel to *AB* and at this distance from it. And place these weights all on this line, anywhere along it.

Notice that they are still balanced by the same weight on the other side of *AB* in the same position as before.

We see now that, if the masses of any number of particles are known, and their co-ordinates, with reference to two axes, the co-ordinates of the C.G. of the particles can at once be found. This is illustrated in the following example.

EXAMPLE.—AOB, COD are two straight lines in a plane at right angles to each other. Four particles of masses 2, 2, 3, 3, are situated in the angles BOC, COA, AOD, DOB respectively, and so that the distances of the particles from the lines CD, AB have the following pairs of values: 2, 3; 3, 2; 2, 3; 3, 2. Where is the Centre of Gravity?

Let the distances of the C.G. from CD towards B, and from AB towards C be x, y.

The entire mass $=10$.

By moments about CD,

$$10x=2 . 2-2 . 3-3 . 2+3 . 3$$
$$=1.$$
$$\therefore\ x=\tfrac{1}{10}.$$

By moments about AB,

$$10y=2 . 3+2 . 2-3 . 3-3 . 2$$
$$=-5.$$
$$\therefore\ y=-\tfrac{5}{10}=-\tfrac{1}{2}.$$

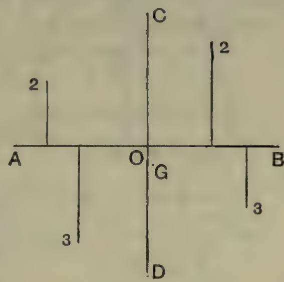

FIG. 91.

The C.G. is **at G, so that its distance from CD towards B is $\frac{1}{10}$, and its distance from AB towards D is $\frac{1}{2}$.**

Notice that we have supposed y to be the distance of G from AB measured towards C. The negative value of y denotes that G is on the side of AB away from C. In fact, since the sum of the numerical values of the moments of the two masses 3, 3 about AB is greater than that for the two masses 2, 2, the Centre of Gravity must be on the same side of AB as the masses 3, 3.

Avoiding the difficulty of Signs.—To avoid the difficulty of signs we may reason as follows:

Sum of moments of the masses 3, 3 about AB is

$$3 . 3+3 . 2=15.$$

Sum of moments of the masses 2, 2 about AB is

$$2 . 3+2 . 2=10.$$

$\therefore$ resultant moment about AB is in same sense as that of moments of the masses 3, 3, and is equal to 5.

$\therefore$ C.G. is on side of AB towards D, at distance $\frac{5}{10}=\frac{1}{2}$.

Similarly we may determine on which side of CD the C.G. is, and its distance from CD.

EXAMPLE.—*ABCD* is a figure composed of four uniform bars *AB*, *BC*, *CD*, *DA*, of lengths 4, 4, 6, 4, and of weights proportional to their lengths. *AB* and *CD* are parallel. Find the position of the Centre of Gravity of the figure.

The weights of the bars may be taken as 4, 4, 6, 4, acting at *K*, *M*, *L*, *N*, their middle points.

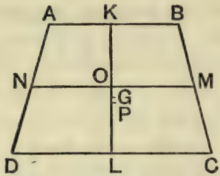

FIG. 92.

Let the straight lines *KL*, *MN* meet in *O*.

4 at *M* and 4 at *N* are equivalent to 8 at *O*.

4 at *K* and 6 at *L* are equivalent to 10 at *P*, where

$$\frac{LP}{PK}=\frac{4}{6}=\frac{2}{3}.$$

8 at *O* and 10 at *P* are equivalent to 18 at *G*, where

$$\frac{OG}{GP}=\frac{10}{8}=\frac{5}{4}.$$

$$\because \frac{LP}{PK}=\frac{2}{3},$$

$$\frac{LP}{LK}=\frac{2}{5},$$

$$LP=\tfrac{2}{5}LK.$$

$$\therefore OP=\tfrac{1}{2}LK-\tfrac{2}{5}LK=\tfrac{1}{10}LK.$$

$$\because \frac{OG}{GP}=\frac{5}{4},$$

$$\frac{OG}{OP}=\frac{5}{9}.$$

$$OG=\tfrac{5}{9}\,.\,OP=\tfrac{1}{18}LK.$$

$$\therefore LG=\tfrac{1}{2}LK-\tfrac{1}{18}LK=\tfrac{4}{9}LK.$$

The C.G. is on the line *LK* joining the middle points of *AB* and *DC*, $\frac{4}{9}$ of the length from *L*.

This result may also be obtained in the following manner, illustrating the advantage of the method of moments.

The distance of each of the points *N* and *M* from *CD* is $\frac{1}{2}LK$.

Let x be the distance of the C.G. from CD.

Then, by moments about CD,

$$(4+4+4+6)x=4 \,.\, LK+4 \,.\, \tfrac{1}{2}LK+4 \,.\, \tfrac{1}{2}LK,$$

$$18x=8LK.$$

$$x=\tfrac{4}{9}\,LK.$$

A body of known Weight and known Centre of Gravity has a piece of known Weight and known Centre of Gravity removed. To find the Centre of Gravity of the remainder.

Let **W** be the weight of the whole body, G its C.G.

Let **w** be the weight of the piece removed, H its C.G.

The weight of the remainder is $\mathbf{W}-\mathbf{w}$.

Call its C.G. K.

Then, since the body is made of the two pieces of weights **w** and $\mathbf{W}-\mathbf{w}$;

$\therefore$ the forces **w** at H and $\mathbf{W}-\mathbf{w}$ at K have for resultant **W** at G.

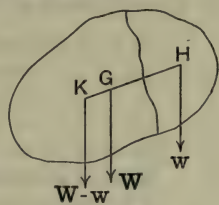

Fig. 93.

From this we infer :

(1) That G is on the straight line HK, and between H and K.
$\therefore$ K is on HG produced.

(2) That $(\mathbf{W}-\mathbf{w})\,KG=\mathbf{w}\,.\,HG$.

$$\therefore \quad KG=\frac{\mathbf{w}}{\mathbf{W}-\mathbf{w}}\,.\,HG.$$

Hence we have the rule.

Join HG, and produce it to K, so that

$$KG=\frac{\mathbf{w}}{\mathbf{W}-\mathbf{w}}\,.\,HG.$$

Then K is the required C.G.

In any particular example, however, it is recommended not to rely on remembering the *rule*, but to work it out by the *method* that has just been used.

Example.—A circular metal disc has a circular portion, of diameter equal to the radius of the disc, bored out, and so that the edge of the piece removed passes through the centre of the disc. Where is the C.G. of the remainder?

Since the areas of circles are proportional to the squares of their diameters, the weights of the disc and the piece removed may be taken as 4 and 1.

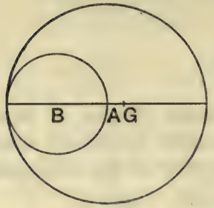

Fig. 94.

$\therefore$ weight of piece left is 3.

C.G.s of disc and piece removed are the centres A and B.

Since the disc is made up of the piece removed and the piece left,

$\therefore$ A lies between B and G, the required C.G.,

$\therefore$ G is in BA produced.

And since the weights 1 at B and 3 at G are equivalent to 4 at A;

$$\therefore\ 3 \,.\, AG = 1 \,.\, AB;$$

$$\therefore\ AG = \frac{AB}{3}.$$

The C.G. of the remainder is **on the line joining the centres at a distance from the centre of the disc of $\frac{1}{6}$ of its radius.**

The C.G. of a regular body, or a body with a known C.G., from which a regular portion has been removed, may sometimes be more conveniently found by an extended application of the principle of moments, as will now be shown.

Consider again the C.G. of the portion of a body of weight **W** which is left when a portion of weight **w** is removed.

The resultant of the weights of all the particles of the entire body is a force **W** acting vertically downwards at G.

By removing the portion of weight **w** we remove a set of forces having resultant **w** acting vertically downwards at H.

This is the same thing as inserting a force **w** acting vertically upwards at H.

Then the weight of the remainder is the resultant of the two forces, **W** vertically downwards at G, **w** vertically upwards at H.

This resultant is $\mathbf{W}-\mathbf{w}$ vertically downwards at a point on HG produced, at K say.

By moments about G we have

$$(\mathbf{W}-\mathbf{w})KG = w \,.\, HG.$$

$$\therefore\ KG = \frac{\mathbf{w}}{\mathbf{W}-\mathbf{w}} \,.\, HG.$$

The following question will illustrate the application of this method.

EXAMPLE.—A cubical block of iron, 10 inches edge, has a piece removed, $8 \times 8 \times 9$ inches in dimensions, so as to leave an open box with sides of thickness 1 inch. The cavity is then filled up with wood. The densities of the iron and wood are as 10 and 1. Find the C.G. of the body formed.

[We shall first solve the question in the more direct, but longer, way; and then show how it may be done by using the principle above explained.]

Let AB be the straight line drawn from the middle point of the open face of the wood at right angles to this and the opposite face.

Let H be the C.G. of the entire block of iron;

L that of the iron left;

K that of the wood and of the iron removed.

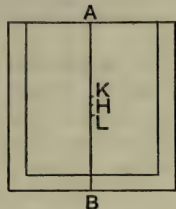

FIG. 95.

$$AH=5'', \; AK=4\tfrac{1}{2}''.$$

Volume of entire block of iron $=1000$ cubic inches,
Volume of piece removed $=8.8.9=576$ cubic inches,
Volume of piece left $=424$ cubic inches,
Volume of wood $=576$ cubic inches.

Dividing by 8, these volumes are proportional to 125, 72, 53, 72.

We may take for the weights ($\because$ the iron is 10 times as heavy as the wood) 1250, 720, 530, 72.

To find L, since the weights 720 at K and 530 at L have resultant at H,

$$530 \,.\, HL=720 \,.\, HK=720 \,.\, \tfrac{1}{2}''.$$

$$HL=\tfrac{36}{53}''.$$

$$AL=5\tfrac{36}{53}''.$$

Let x inches be the distance of the required C.G., G, from

A. Then by moments about *A*, since the resultant of 530 at *L* and 72 at *K* is at distance *x* inches from *A*,

$$(530+72)x = 530 \,.\, 5\tfrac{36}{53} + 72 \,.\, 4\tfrac{1}{2}.$$

$$602x = 3334.$$

$$x = 5\tfrac{162}{301}.$$

The C.G. is **on AB at $5\tfrac{162}{301}$ ins. from A.**

Alternative Method.—The weight of the body is the resultant of the weights of the entire block of iron at *H*, and of the wood at *K*, and a force equal to the weight of the iron removed acting vertically upwards at *K*.

$\therefore$ by moments about *A*,

$$(10 \,.\, 1000 + 1 \,.\, 576 - 10 \,.\, 576)x$$
$$= 10 \,.\, 1000 \,.\, 5 + 576 \,.\, 4\tfrac{1}{2} - 10 \,.\, 576 \,.\, 4\tfrac{1}{2};$$

$$(10 \,.\, 125 - 9 \,.\, 72)x = 10 \,.\, 125 \,.\, 5 - 9 \,.\, 72 \,.\, 4\tfrac{1}{2};$$

$$602x = 3334;$$

$$x = 5\tfrac{162}{301}.$$

Exercises VI. a.

1. *ABCD* is a square. Particles of weights 1, 2, 3, 4 are placed at *A*, *B*, *C*, *D*. Find the distances of their C.G. from *AB* and *AD*.

2. *ABCD* is a square, *O* the intersection of diagonals. *E*, *F*, *G*, *H* are the middle points of *OA*, *OB*, *OC*, *OD*. $OE = a$. Masses 1, 3, 5, 7 are placed at *A*, *B*, *C*, *D*, and 2, 4, 6, 8 at *E*, *F*, *G*, *H*. Find the C.G. and its distances from *AC* and *BD*.

3. Equal masses are placed at all the angles but one of a regular hexagon. Where is their C.G.?

4. *AB*, *BC* are two rods of the same cross-section, of materials whose densities are as 5 and 7. Their lengths are 4 and 6. Find their C.G.

5. Three cylindrical rods, each of length 4 inches, having diameters 1, $1\tfrac{1}{4}$, $1\tfrac{1}{2}$ inches, and densities 6, 7·5, 6 are joined in a straight line. Show that their Centre of Gravity is $\tfrac{16}{61}$ inch from the middle point of the middle rod.

6. Three spheres of radii 6, 10, and 4 inches are joined together with their centres in a straight line. That of radius 10 inches is in the middle, and has a density twice as great as each of the others. Where is their Centre of Gravity?

7. ABC is a triangle; D, E, F the middle points of BC, CA, and AB. Show that the C.G. of $BCEF$ is on the straight line DA at a distance $\frac{2}{9}AD$ from D.

8. ABC is a triangle. A straight line is drawn parallel to BC, cutting off $\frac{1}{n}$ of each of the sides AB, AC. Find the Centre of Gravity of the trapezium formed.

9. A square, of side a and weight per unit area w, has a circular portion of diameter $\frac{a}{2}$ removed, the circumference of the circle touching one side of the square; and this is replaced by a piece, of weight per unit area w'. Show that the distance of the C.G. of the body formed from the side in question is

$$\frac{a(32w - \pi w + \pi w')}{4(16w - \pi w + \pi w')},$$

π being the ratio of the circumference of a circle to its diameter.

States of Equilibrium.—Suppose a body to be in equilibrium under the action of any forces. If it is now slightly displaced from its position of equilibrium the forces acting on it will, as a rule, be slightly altered; and the body will, in general, in the new position, not be in equilibrium, although in certain cases it may still remain in equilibrium after the displacement.

One of three things will happen after the displacement.

(1) The body may tend to return to its old position.

(2) The body may tend to move further away from its old position.

(3) The body may remain in equilibrium in the displaced position, not tending to move either one way or the other.

These cases may be illustrated by a simple example.

Suppose a lamina, such as a piece of cardboard, to be carried in a vertical position by means of a peg which passes through a hole in it, and about which it can freely rotate in its own plane. For the lamina to be in equilibrium its centre of gravity and the point of support must be in the same vertical straight line.

If the body is suspended so that its C.G. is vertically *below* the point of support, after a slight displacement it returns to its old position.

If the body is suspended so that its C.G. is vertically *above* the point of support, after a slight displacement it moves further away from its old position.

If the body is suspended so that its C.G. is *at* the point of support, after a slight displacement it remains in the displaced position, moving neither one way nor the other.

The equilibrium of the body is said in these three cases, respectively, to be **stable, unstable,** and **neutral.**

The equilibrium of a body is stable, unstable, *or* neutral, *according as, after a very slight displacement, the body returns to its old position, moves farther away from it, or remains in the displaced position.*

Instances.—A body resting on an extended base, such as a chair or table, or a book lying flat on a table, is, in general, in stable equilibrium.

A body balanced on a point would be in unstable equilibrium. Such a condition it is practically impossible to realize ; if it could for an instant be attained the slightest accidental shake or draught of wind would upset the equilibrium.

If a book is placed upright, so as to stand on its edge, the equilibrium is stable, because a very slight, infinitesimal displacement would not upset it. But we may recognise different *degrees of stability ;* and the book in this position would be in less stable equilibrium than if lying flat on the table.

A uniform sphere resting on a horizontal table is in neutral equilibrium.

It may happen that the state of equilibrium of a body under given circumstances is not the same for all the displacements which it is possible to give it.

Thus, if an egg lies on a table, its equilibrium is stable for a displacement in which one end is raised ; but neutral for a displacement in which it is rolled along, the ends each remaining at the same height throughout the rolling.

A square lamina balanced on one edge on a horizontal surface is (theoretically) in stable equilibrium for one sort of displacement, and in unstable equilibrium for another.

Criterion for State of Equilibrium.—A very important case of equilibrium is that of a body under the action only of its own weight and the reactions of fixed supports, such as fixed smooth surfaces in contact with which it rests. In this case there is an easy criterion for the state of the equilibrium for any displacement, as we shall now show.

The C.G. of a body always tends to fall lower and lower.

If the C.G. is raised by the displacement, after the displacement the C.G. falls, and the body returns to its old position.

If the C.G. is lowered by the displacement, after the displacement it will not rise again, but continue to fall farther away from its old position.

If the C.G. is unaltered in level by the displacement, after the displacement it will tend to move neither way.

Hence the equilibrium of the body is stable, unstable, or neutral, according as the displacement raises its C.G. or lowers it, or does not alter its level.

If a body is capable of being moved in several different ways it often happens that it can receive a displacement for which the equilibrium is partly of one sort and partly of another. Take, for example, the case of a book lying on a table. It may receive a slight displacement consisting partly of sliding it along the table and partly of raising one end of it. If it is then released it will fall back on the table, but not into its original position.

But it should be noticed that this case differs from that of a cylinder lying flat on a smooth horizontal table in an important respect.

The cylinder can be displaced by the application of a *very small force* acting horizontally and at right angles to its axis, in fact, theoretically *by any force, however small.*

To displace the book on the table a force large enough to overcome the friction is required, and a smaller force would not produce displacement.

The book resting on the table is said, as a rule, to be in *stable* equilibrium, and the cylinder to be in *neutral* equilibrium.

EXAMPLE.—ABC is a triangle cut from a board of uniform thickness and density. The angle B is acute. If $AC=b$, $CB=a$, find the greatest value that AB can have, in order that the triangle may be able to stand on CB on a horizontal surface.

Join C to F, the middle point of AB.

Then the C.G. of the triangle is in CF. And for the triangle to be just on the point of falling over BCF must be a right angle, and this gives the greatest length for AB.

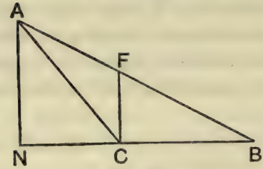

FIG. 96.

Draw AN perpendicular to BC produced.

Then $NC = CB = a$.

$$BA^2 = BN^2 + NA^2$$
$$= (2a)^2 + b^2 - a^2$$
$$= 3a^2 + b^2.$$

The greatest value that BA can have is $\sqrt{3a^2+b^2}$.

EXAMPLE.—Four bricks are placed evenly, one on top of the other on a horizontal surface, each brick projecting in the direction of its length by the same amount beyond the one below it. If the length of each brick is l, find the greatest length by which each brick can project beyond the one below it.

The arrangement must be such that the C.G. of any number of the bricks, counting from the top down, must not lie beyond the edge of the brick immediately below them.

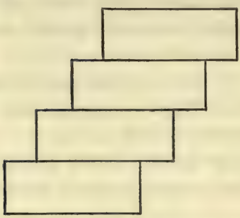
FIG. 97.

Now if this is true for three bricks it is clearly true for two and for one, because, as more bricks are added on, the C.G. is more and more displaced.

Let x be the amount by which each brick overlaps the one below it, when the three top bricks are just on the point of falling over.

Then the C.G. of these three is displaced by $\frac{l}{2}$ from the vertical line through the middle point of the lowest. Thus we have, putting 1 for the weight of each brick,

$$3 \cdot \frac{l}{2} = 1 \cdot x + 1 \cdot 2x + 1 \cdot 3x, \quad 3l = 12x, \quad x = \frac{l}{4}.$$

Thus the required length is $\frac{l}{4}$.

EXAMPLE.—A rectangular block of wood $9 \times 9 \times 12$ ins. stands on one of its small faces on a horizontal plane. Its weight is 25 lbs. What is the smallest thrust at the middle point of one of its top edges that will upset it, supposing that it will turn without sliding about a lower edge?

Let $ABCD$ represent a section of the block, so that G is its C.G., A the middle point of the edge about which it turns, and C the point to which the force is applied.

The moment about A of the upsetting force has to overcome the moment about A of the weight. This moment is greatest at first, when the face AB is horizontal. Thus we must find the upsetting force necessary to counterbalance the weight in the initial position.

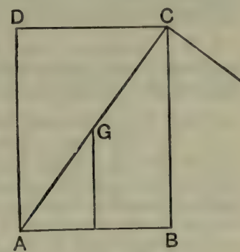

FIG. 98.

Now in order that the upsetting force may be as small as possible it must act so as to produce the greatest possible moment about A. Hence it must be at right angles to AC. If the required force at starting is F pounds' weight, by moments about A,

$$F \,.\, AC = 25 \,.\, \tfrac{1}{2} AB.$$

$$F \,.\, 15 = 25 \,.\, \tfrac{9}{2}.$$

$$F = 7\tfrac{1}{2}.$$

Hence smallest thrust necessary is **just over $7\frac{1}{2}$ lbs.' weight.**

Exercises VI. b.

1. A table with a square top, 2 feet each way, has four legs each $28\frac{1}{2}$ inches long, and set at $2\frac{1}{2}$ inches from the edges. The entire table weighs 20 lbs. Find the least vertical and horizontal forces at an edge necessary to upset it.

2. AC, BC are two light rods, of lengths b and a, hinged at C, and connected by a string AB. Weights W and $2W$ are fastened to A and B. For the arrangement to rest on BC, show that the least length for the string is $\sqrt{2a^2+b^2}$.

3. A cylinder, the diameter of whose base is 5 inches, and height 8 inches, stands on a horizontal surface. How high can one side of the base be raised without causing the cylinder to fall over?

4. If n bricks are piled one on top of the other, with their lengths horizontal, each projecting by the same amount in the direction of the lengths beyond the one below it, show that the greatest value that this projection can have is $\frac{1}{n}$ of the length of each brick.

5. A circular table stands on three symmetrically-placed legs, their lower ends being all just vertically beneath the edge of the table. Show that a body of less weight than that of the table may be placed anywhere on it without upsetting it, but that a body of greater weight can be placed so as to upset it.

6. A lead pencil AB, 6 inches long, with a weight of 6 oz. hanging from a point $\frac{1}{4}$-inch from the end A, is kept in a horizontal position by a thumb and forefinger. The forefinger is above the pencil at a point $\frac{1}{4}$-inch from the end B, and the thumb under the pencil at a point 1 inch from the end B. Neglecting the weight of the pencil, find the pressures on the thumb and forefinger. (Coll. Precep., 1898.)

7. A uniform beam AB, of length 12 feet and weight 10 lbs., is suspended horizontally by two vertical cords attached to the ends, and a weight of 20 lbs. is placed on the beam. Find the position of this weight that the tension of one of the cords may be exactly double that of the other. (Coll. Precep., 1897.)

8. Three solid cubes of granite, whose edges are 3 feet, 2 feet, and 1 foot long respectively, are piled on a horizontal platform. The largest is at the bottom of the pile, the smallest at the top, and each face of each cube is either horizontal or vertical. At what height above the platform is their Centre of Gravity? (Coll. Precep., 1898.)

9. Mention the chief properties of the Centre of Gravity of a body.

An isosceles right-angled triangle is constructed on the upper side of a square as hypothenuse. Find the Centre of Gravity of the whole figure.

10. Find, geometrically, the Centre of Gravity of a heavy bar 10 feet long, bent so as to form an angle 4 feet from one end. (Coll. Precep., 1898.)

11. A lever is to be cut from a bar weighing 3 lbs. per foot. What must be its length that it may balance about a point 2 feet from one end when weighted at this end with 50 lbs.? (Coll. Precep., 1898.)

12. Masses of 1, 2, 5, and 10 grams are placed in this order round the corners of a rectangle whose sides are 12 cm. and 3 cm. long, the 1 and 2 grams being at the ends of a side 12 cm. long. Determine the position of the Centre of Mass of the system. (London Matric., 1898.)

13. The Centre of Gravity of a rod 4 feet long, weighing 6 lbs., is 1 foot from one end of the rod. At the other end a weight of 4 lbs. is placed; find where the rod must be supported so that it rests in a horizontal position. (Oxford Locals, 1897.)

14. Define the Centre of Gravity of a body. Masses of 1, 2, 3 lbs. respectively are placed at the vertices of a triangle ABC; find their Centre of Gravity. (Oxford Locals, 1898.)

15. A uniform rod AB, 2 feet long, weighing 12 lbs., can turn freely in a vertical plane about an axis at A. Calculate the force which must be applied at B, at right angles to AB, to keep the rod in equilibrium at an angle of 45° to the vertical. (Oxford Locals, 1898.)

16. A uniform board 1 foot square has a weight of 1 lb. fastened to each of two adjacent corners, and a weight of 2 lbs. to each of the other two. Find its Centre of Gravity. (Camb. Jr. Loc., Stat. Dyn. and Hydro., 1896.)

17. If from a uniform circular sheet of lead, 1 foot in radius, a round hole 1 inch in radius be punched out, find the Centre of Gravity of the remaining portion when the centres of the disc and the hole are 7 inches apart. (Camb. Jr. Loc., Mech., 1897.)

18. A uniform ladder 30 feet long, weighing 20 stone, rests with one end on the ground and the other end against a smooth vertical wall, with which it makes an angle of 30°. A man weighing 10 stone climbs 24 feet up the ladder. Calculate the horizontal and vertical components of the force exerted by the ground on the foot of the ladder. (Camb. Jr. Loc., Mech., 1898.)

19. Particles whose masses are 2, 5, 2, 3 are placed in order at the angular points of a square; show in a diagram the position of their Centre of Gravity, and find its distance from the particle whose mass is 5. (Science and Art, 1897.)

20. Two smooth cylinders are held firmly with their axes horizontal and parallel, and with their centres 5 inches apart; the radius of each is an inch long; a third cylinder, weight 100 lbs., and radius 5 inches, rests on the two fixed cylinders, with its axis parallel to

their axes; find the pressure it exerts on each of the small cylinders. (Science and Art, 1897.)

21. Two uniform bars AB, BC, of lengths 2 feet and 1 foot, are rigidly connected at B so as to form an angle; find the distance of their common Centre of Gravity from the middle point of AB. (Science and Art, 1899.)

Summary.

If the masses of any particles in a plane are known, and their distances from two straight lines in different directions, their C.G. can be found by taking moments about these straight lines.

If the mass and C.G. of a body are known, and the mass and C.G. of a piece removed are known, the C.G. of the remainder may be found by noticing that the piece removed and the piece left would balance about the C.G. of the whole.

States of Equilibrium.—A body is said to be in stable, unstable, or neutral equilibrium, according as when it is very slightly displaced from its position of equilibrium it tends to return, to go further away, or to do neither.

A heavy body is in stable, unstable, or neutral equilibrium for a given kind of displacement, according as the displacement raises or lowers its C.G. or does neither.

CHAPTER VII.

STATES OF MATTER. ELASTICITIES.

Different states of Matter.—There are three different physical states in which matter may exist: the solid, liquid, and gaseous states.

A *solid body* is one which, in general, offers considerable resistance to the application of forces tending to change its shape. Thus, if forces act on a rod which tend to bend it, the rod will yield to a certain extent and no further, unless the forces are great enough to break it.

A *liquid*, in general, yields more readily to forces tending to change its shape. Thus, water yields with great readiness, treacle less readily, and pitch less readily still. The last two liquids are said to be *viscous.*

It is difficult to draw the line between solids and liquids. There are some soft solids which readily change their shapes, and, on the other hand, there are viscous liquids which only change their shapes with great reluctance. But the distinction is this.

A solid will not continue to change its shape indefinitely under the continued application of any force, however small, but a liquid will do so.

A *gas*, such as air, is a body which readily undergoes changes of volume by varying the pressure to which it is subjected. Thus, by increasing the pressure acting on a quantity of air, it readily diminishes in volume. Water, or any other liquid, can also be compressed by subjecting it to increased pressure, but the diminution in volume, in this case, is very small.

The distinction between a liquid and a gas is this—

By diminishing the pressure on a liquid its volume will only increase to a certain limit. By continually diminishing the pressure on a gas, its volume will continually increase, and occupy any assigned space, however large.

In general, in questions on the Mechanics of Solids, we have not to consider deformations produced in bodies by the forces acting on them, but to regard the bodies as quite undeformable. But in certain cases we have to consider deformations.

Elasticity.—If a body is subject to the action of forces tending to deform it in any way, the property by which it resists deformation is called Elasticity.

A solid body possesses various elasticities, according to the various sorts of deformation which it may be forced to undergo.

Thus, a rod may be held at one end and stretched, that is, it may undergo a longitudinal deformation.

It may be subjected to equal pressure on all sides, so that it is made to undergo a diminution in bulk.

It may be held at one end while the other end is twisted, so that it is made to undergo a torsional deformation.

Corresponding to these three sorts of deformation, the rod has longitudinal, volume, and torsional elasticities.

Young's Modulus.—The longitudinal elasticity is also called, more commonly, the *Young's modulus* for the material. We will consider more in detail the exact meaning of this quantity, and the way to measure it.

Suppose a straight rod of uniform cross-sectional area, such as a long cylinder, to be firmly fixed at one end and stretched by a force applied at the other end. By the cross-sectional area is meant the area of the section, or of either of the two faces that would be exposed by making a cut across the rod at right angles to its length.

Now, it is clear that if a force of a certain magnitude stretches the rod by a certain amount, twice as much force would be required for a rod made of the same material and of the same length, but of twice the cross-sectional area of the given one. Three times the force is required to produce the same stretching in a rod of three times the cross-sectional area, and so on. And in general the tendency of a force to produce stretching, and the amount of stretching produced in a given length of rod, will depend, not only on the force, but on the amount of force acting across each unit area of cross-section.

The force acting per unit area of cross-section of the rod, or the fraction

$$\frac{\textit{Force}}{\textit{Area of cross-section}}$$

is called the **stress** in the rod.

If a rod of a given length is stretched by a certain amount, the amount of stretching produced in each unit of its length depends, not only on the whole stretching, but on the original length.

The stretching per unit of length, or the fraction

$$\frac{\textit{Stretching}}{\textit{Original length}}$$

is called the **strain** in the rod.

Stresses and strains in materials may be of various sorts, and in any case the fraction *stress ÷ strain* is called the *elasticity* of the material for the particular sort of deformation produced.

Thus we have, in general,

$$\textbf{Elasticity} = \frac{\textbf{Stress}}{\textbf{Strain}}.$$

In the particular case of stress and strain considered, the elasticity in question is the longitudinal elasticity, or the Young's modulus.

Limit of Elasticity.—Experiment shows that for a given material the stress and the strain are always proportional to each other, provided these are both of moderate amount. Thus, if a rubber band is stretched, the elongation is proportional to the stretching force; and if rubber bands of various sizes, but all made of the same material, are used, it will be found that the elongation per unit of original length is always proportional to the force per unit area of cross-section.

It is only when the force applied is so great as to produce a permanent deformation, so that the body does not return to its original dimensions when the force is removed, that the stress and strain are no longer proportional.

Thus, as long as we keep to moderate forces, not producing permanent deformation, the definition given for elasticity leads to a constant value for this quantity.

Suppose we have a rod of length l and cross-sectional area s, and suppose that a stretching force $\mathbf{p}$ stretches the rod by a length l'.

In this case, $\text{stress} = \frac{\mathbf{p}}{s}$;

$$\text{strain} = \frac{l'}{l}.$$

Let the longitudinal elasticity, or Young's modulus, be m.

Then $$m = \frac{\mathbf{p}}{s} \div \frac{l'}{l} = \frac{\mathbf{p}l}{sl'}.$$

Compression.—If, instead of a stretching force **p**, a thrust **p** acts at the end of the rod so as to *compress* it in the direction of its length and make it shorter, then, as long as no bending or buckling is caused, the rod will be shortened by the amount l', that is, by an amount equal to the lengthening caused by a pull **p**. In other words, if the stress is reversed but kept of the same magnitude, the strain will be reversed, but will be of the same magnitude.

In the equation $$m = \frac{\mathbf{p}l}{sl'},$$

since $\frac{l}{l'}$ is simply a ratio, m is a quantity of the same sort as $\frac{\mathbf{p}}{s}$; that is, it is to be expressed in units of force per unit of area; for example, in pounds' weight per square inch.

p, s, and m must all be expressed in consistent units. Thus, if m is in pounds' weight per square inch, **p** must be in pounds' weight and s in square inches. But l and l' may be expressed in terms of any unit of length, provided the same unit is used for both, for it is merely the ratio of these two quantities that we have to deal with, and this will be the same if they are expressed in inches, feet, centimetres, or anything else.

EXPERIMENT 21.—(1) Take a uniform rubber band and suspend it by one end. Hang various weights on the other end, such as 10, 20, 30, 40, 50 grams. Measure the elongations produced. These should be proportional to the weights used.

(2) Shorten the band up so as to use only half its length. It may be cut in two or suspended by its middle. Find the elongations produced by the same weights. These should be half as great as in the first case.

(3) Attach the two halves side by side, tying the ends together. Find the elongations produced by the same weights. These should be half as great as in the second case.

EXPERIMENT 22.—Find the elongation produced by a certain weight in a rubber band, say 50 grams. Measure the unstretched length of the band; and by carefully measuring its breadth and thickness in its natural condition, determine its cross-sectional area. Thus find the Young's modulus of the band.

EXAMPLE.—What is the diameter of a steel wire when a weight of 1 cwt. stretches 100 yards of it $\frac{1}{4}$ of an inch? (Young's modulus for steel = 31,000,000 lbs.' wt. per square inch.)

Let diameter be d inches.

$$\text{Area of cross-section} = \frac{1}{4}\pi d^2 = \frac{11d^2}{14} \text{ sq. ins.}$$

$$\text{Stress} = 112 \div \frac{11d^2}{14} \text{ lbs.' wt. per sq. in.}$$

$$= \frac{112 \cdot 14}{11d^2} \text{ lbs.' wt. per sq. in.}$$

$$\text{Strain} = \frac{1}{48} \div 300 = \frac{1}{14400},$$

$$\therefore\ 31{,}000{,}000 = \frac{112 \cdot 14}{11d^2} \div \frac{1}{14400}.$$

$$d^2 = \frac{112 \cdot 14 \cdot 14400}{31000000} = \frac{32 \cdot 7^2 \cdot 12^2}{310000},$$

$$d = \frac{7 \cdot 12}{100}\sqrt{\frac{32}{31}} = {\cdot}853.$$

$$\therefore\ \textbf{diameter} = {\cdot}853 \textbf{ ins.}$$

Exercises VII.

1. What weight will stretch a copper wire 10 feet long and $\frac{1}{4}$-inch diameter by $\frac{1}{4}$-inch?

Mod. = 18,000,000 lbs.' wt. per sq. inch. $\pi = \frac{22}{7}$.

2. Find Young's modulus for a material, a bar of which 1 yard long, with a cross-sectional area of 4 square inches, is compressed $\frac{1}{10}$-inch by a force of 100 lbs.' wt.

3. A uniform bar 1 foot long is subjected to a tension of 1 ton per square inch of its sectional area. What is the nature of the resistance which it offers to elongation?

If the elongation be $\frac{1}{10}$ of an inch, find the modulus of elasticity (Science and Art, 1898.)

Bending.—Suppose a beam or rod to be placed horizontally, and to be firmly fixed at one end; then let it be loaded with a weight at the other end. If this is not great enough to break or permanently deform the beam, it will produce an amount of bending, or of deviation of this end from its original position, proportional to itself.

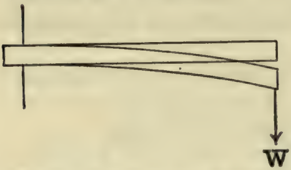

FIG. 99.—Bending of rod fixed at one end.

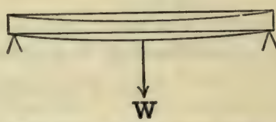

FIG. 100.—Bending of rod supported at both ends.

If the beam rests on supports at its ends, and carries a load between the supports, the load will depress the point to which it is attached by an amount proportional to itself.

In each of these cases the beam bends because some parts of it become stretched and others compressed. In the first case the upper layers of the beam are stretched and the lower ones compressed, and in the second case the reverse takes place. Hence, in any case of bending of a beam the same elasticity is concerned as in simple stretching or compression, that is, the Young's modulus.

EXPERIMENT 23.—Take a long flexible lath, about 3 feet long, 1 inch broad, and $\frac{1}{4}$-inch thick would be suitable dimensions, but these may be considerably varied. Fix one end firmly in a vice, or by laying about 6 inches of it on the edge of a table and placing a heavy weight on it, so that the lath is horizontal.

Load the other end with various weights, such as 10, 20, 30, etc., grams. These may be carried by a string tied to the end, or they may be carefully placed in turn so that their centres come over a mark made across the lath near its end.

Notice the depressions of the end that are produced by the weights. These should be proportional to the weights.

EXPERIMENT 24.—Support the lath horizontally with its two ends resting on two supports, such as two blocks of wood.

Place in turn various weights at the middle of the lath, and measure the depressions of the middle which are produced by the weights. These should be proportional to the weights.

Hydrostatic Pressure.—If a body is subjected to pressure on all sides which is the same per unit of area of its surface everywhere, its volume will be diminished. Such a state of pressure may be produced by placing the body in fluid contained in a suitable vessel, and forcibly compressing the fluid. The number of units of force acting on each unit of area of the body is called the *hydrostatic pressure.* Let this be **P**.

If the original volume of the body is V, and this is reduced by an amount v, the diminution of volume per unit of original volume is $v \div V$.

In this case,

$$\text{Stress} = \mathbf{P}\ ; \quad \text{Strain} = \frac{v}{V}.$$

The corresponding elasticity, or the volume elasticity of the body is

$$\mathbf{P} \div \frac{v}{V}, \text{ or } \frac{\mathbf{P}V}{v}.$$

This elasticity may be regarded as the measure of the resistance which the body offers to compression.

The reciprocal of the elasticity, or the diminution in volume per unit of original volume, divided by the hydrostatic pressure, is called the *compressibility*; and this may be regarded as the measure of the readiness of the body to yield to compression.

Twisting.—If a long cylindrical rod or wire be firmly fixed at one end, it may be twisted by attaching an arm like ACB to the other end C, so that $AC = CB$, and applying equal forces at A and B, so that these act at right angles to AB, and in a plane at right angles to the wire. These forces form a couple of moment $\mathbf{P} \cdot AB$ in a plane at right angles to the wire.

The amount of torsion produced in the wire, that is, the angle through which the end C is twisted, will depend on the twisting couple and on the material and dimensions of the wire.

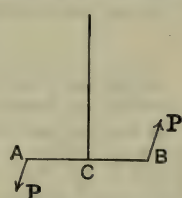

FIG. 101.—Torsion produced in wire by twisting couple.

If the wire is of length l and radius r, and if the moment of the twisting couple is M, it is found that the torsion in the wire is proportional to M and l, and inversely proportional to r^4.

Thus the torsion is

$$\frac{CMl}{r^4},$$

where C is a constant depending on the material of the wire.

Summary.

There are three different states in which matter may exist, the solid, liquid, and gaseous states.

A **solid** will not go on yielding to the continuous application of very small forces tending to change its shape or size.

A **liquid** will yield indefinitely to the application of forces, however small, tending to change its shape; but its volume will not increase beyond a definite amount, however small the pressure to which it may be subjected.

A liquid which only slowly yields to the application of force is said to be **viscous**.

A **gas** will occupy any volume whatever, however great, if the pressure to which it is subjected be continually diminished.

Elasticity is a property of a body, in virtue of which it resists the application of forces tending to deform it.

A body may have various elasticities according to the various sorts of deformation that it can undergo.

Stretching.—If a straight rod of uniform cross-section is stretched, then

The *strain* is the stretching per unit length;

The *stress* is the stretching force per unit of area of cross-section.

In any case of deformation,

$$\textbf{Elasticity} = \frac{\textbf{stress}}{\textbf{strain}}.$$

In the case mentioned, the particular elasticity is the longitudinal elasticity, or **Young's modulus**.

If a rod of uniform cross-section be *compressed* in the direction of its length, then, so long as there is no buckling, the diminution of length is the same as the increase in length that an equal stretching force would produce.

Hence questions of such compression may be treated by the application of the same elasticity, Young's modulus, as is used in questions of elongation.

If a rod or beam is supported horizontally, either by being firmly fixed at one end, or by resting on two supports at the ends, and be bent by means of weights, the amount of bending is proportional to the bending weight, so long as this is not so great as to permanently deform the beam. The elasticity concerned in this case is Young's modulus.

Compression in bulk.—If a body is subjected to hydrostatic pressure which diminishes the bulk, the corresponding elasticity, or

$$\textbf{volume elasticity} = \frac{\textbf{pressure per unit area}}{\textbf{decrease in bulk per unit of original bulk}}.$$

The reciprocal of this is called the **compressibility**.

Torsion.—If a uniform cylindrical rod be held at one end, and the other end be twisted, the amount of torsion produced depends on the material and on the dimensions of the rod, as well as on the moment of the twisting couple.

It is proportional to the moment of the couple and the length of the rod, and inversely proportional to the fourth power of the radius of the cross-section of the rod.

CHAPTER VIII.

WORK. POWER. ENERGY.

Work.—Work is done when a working agent, such as a man, a horse, or a steam engine, overcomes a resistance through a distance.

Thus, if a man raises a body, thus overcoming the resistance of its weight, through a height, he does work. But in merely sustaining the weight of the body and keeping it still from falling, no matter for how long he holds it up, the man does no work. Again, if a horse draws a carriage along a road he does work.

Suppose that several men push a heavy railway van and move it. One man alone could not overcome the resistance, and so by pushing alone he would do no work, for he would not move the van. All the men together do work. And we must suppose that each man does a portion of the work depending on the portion of the entire force which he contributes.

Thus, we may say that an agent does work *when it exerts a force through a distance.*

Measurement of Work.—Consider next how the work done by an agent is to be measured.

Take the case of a horse pulling a carriage along a road against a definite resistance to motion, and, consequently, exerting a force equal to the resistance in order to overcome it.

In pulling the carriage along a definite distance a certain amount of work is done. In pulling the carriage along another equal distance an equal amount of work is done. If the distance is made three times as great the work is clearly three times as great as at first, and so on.

Again, in pulling two exactly similar carriages, so that twice as much force is necessary, through the given distance the work is twice as great as for one, and so on.

Hence the work done is proportional to the distance and proportional to the force.

Thus it is proportional to the product,

force × distance ;

and it will be consistent to define it as equal to this product.

A force, or the agent which exerts the force, may do work when the point of application of the force is displaced, but not in the direction in which the force acts.

In the case of the men pushing the van, one man may push obliquely, but he still helps to move it on in the direction in which it actually moves, and so he does some work. If he pushes in a direction straight across that of the motion, or at right angles to the lines, he neither helps nor hinders the motion, and hence he does no work.

Resolve the force exerted by the man into two components, one in the direction of the motion, the other at right angles to it. The second of these does no work. The work done by the man is the product of the resolved part of the force in the direction of the motion and the distance by which the van is displaced.

In general, suppose a force **F** to act in the direction AC; and let its point of application undergo displacement from A to B.

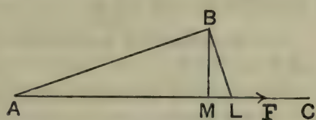

FIG. 102.—Work done by force which is inclined to displacement.

Draw BM, BL at right angles to AC, AB.

Then, if **F** is resolved along and at right angles to AB, its component along AB is, by the triangle ABL,

$$\mathbf{F}\cdot\frac{AB}{AL}=\mathbf{F}\cdot\frac{AM}{AB}.$$

$$\therefore \text{ the work done by } \mathbf{F}=\mathbf{F}\cdot\frac{AM}{AB}\cdot AB=\mathbf{F}\cdot AM.$$

Again, we may suppose the displacement AB to be made up of, or compounded of, two displacements AM, MB; and since the work done by **F** is **F** . AM, we may say that it is the product of the force and the resolved part of the displacement in the direction of the force.

Negative Work.—It is sometimes necessary to consider work done as negative.

Suppose that, in the case of the men pushing the van, one man pushes so as to hinder the motion, that is, either directly obliquely or backwards. On account of this, when we consider the whole effect produced, something must be deducted. The work done by this man is a negative quantity.

The resolved part of the force which he exerts in the direction of the motion is a negative force; and the work he does, measured as before, by the product of this component of force and the displacement, is negative.

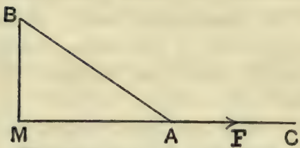

FIG. 103.—Negative work done by force.

In the figure, suppose that AB, the direction of the displacement, makes an obtuse angle with AC, the direction of the force.

The resolved part of **F** in the direction of AB must be counted as $-\mathbf{F} \cdot \frac{AM}{AB}$.

Or, the resolved part of the displacement AB in the direction of **F** is $-AM$.

From either of these we see that the work done by **F** is $-\mathbf{F} \cdot AM$.

If AB is at right angles to **F**, $AM=0$, and the work is 0.

We may notice that the work is positive, zero, or negative, according as the angle between the force and the displacement is an acute angle, a right angle, or an obtuse angle.

General Definition.—We may now give the following general definition.

If the point of application of a constant force undergoes a rectilinear displacement the work done by the force is:

(a) *The product of the displacement and the resolved part of the force in the direction of the displacement; or*

(b) *The product of the force and the resolved part of the displacement in the direction of the force.*

The distance AM, in the above figures, or the resolved part of the displacement in the direction of the force, is sometimes called the distance through which the force acts. And the work done by the force is then called *the product of the force and the distance through which it acts.*

Unit of Work.—The unit of work is, in general, the work done when unit of force acts through unit distance.

Thus, if we use the lb.'s weight and the foot as units of force and length, we get as unit of work the work done by a lb.'s weight acting through a foot. This is a very important unit of work in practical questions. It is called the **foot-pound**, and it is the British engineers' unit of work.

Other units of work are derived from other methods of measuring force and length. The work done when a kilogram's weight acts through a metre is called the **kilogram-metre.**

If PQ is any line, straight or curved, joining the two points P, Q, and PM, QN are perpendiculars on a straight line XY, then MN is called the projection of PQ on XY.

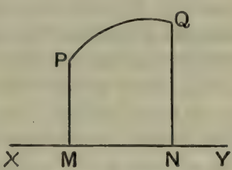

Fig 104.—Projection of a line.

In the case we have considered we may say that the work done by the force **F** as its point of application is moved from A to B is the product of **F** and the projection of A on the constant direction of **F**, or any fixed straight line to which **F** is always parallel.

If a force **F**, constant in direction and magnitude, acts while its point of application moves from A to B along any path joining A to B, the work done is the product of **F** and the projection of AB on the direction of **F**.

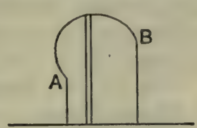

Fig. 105.

For AB may be supposed to be divided in indefinitely small elements, all of which are straight. And the work is the product of **F** and the algebraical sum of the projections of these elements, that is, the product of **F** and the projection of AB.

Power, in Mechanics, means rate of doing work.

The word is sometimes used in other senses, especially to denote the force applied to a machine, or mechanical appliance, in order to do work by means of the machine. But such a use of the word should be avoided, its true meaning being, as stated, *rate of doing work.*

It is usual to speak of the power of the agent, man, horse, steam-engine, or otherwise, which is performing the work, or the power at which the agent is, in the given case, working.

Thus, if an agent is exerting a force of 12 lbs.' weight through 8 feet in every second, we may say that the power of the agent or the power at which it is working is 96 foot-pounds per second.

The rate of working at 1 foot-pound per second is an inconveniently small rate or power with which to compare other powers. For if we were to express the powers of large steam engines in foot-pounds per second, the numbers we should have to use to specify the powers would be very large ones indeed. For this reason a practical unit of power is employed called the **horse-power,** which may be defined as the rate of working at 33,000 foot-pounds per minute or 550 foot-pounds per second.

The horse-power was determined by Watt. It was intended, as its name implies, to denote the rate at which a horse can work; but it is an overestimate of the power of an average horse working in ordinary circumstances.

Suppose a man is raising a body weighing 25 lbs. through 2 feet in each second. He works at the rate of 50 foot-pounds per second, that is at the rate of $\frac{50}{550}$ or $\frac{1}{11}$ of a horse-power.

The letters H.P. are used as an abbreviation of the phrase horse-power. They are often used, in a rather different sense, to mean "the number of horse-power in a given rate of working."

Thus we may say, an engine works at 2 H.P.; or the H.P. of an engine is 2.

Both of these mean the same thing.

Cheval-Vapeur.—The French practical unit of power corresponding to the English horse-power is the *cheval-vapeur.* This is the rate of working at 75 kilogram-metres per second.

The cheval-vapeur is a little smaller than the horse-power, being about 542 foot-pounds per second.

Energy.—The word *energy* is equivalent to *work.* It is generally used as applying to the work which can be got out of an agent, or which an agent is capable of doing, under given circumstances.

Thus we may speak of the work done by an engine, or other agent, as the energy supplied by it. We may speak of the energy given out by a water-fall in a given time. And we may say that the power of an engine means the rate at which it develops energy.

Work done against Gravity.—To raise a body weighing **W** lbs. from a given position to another, which is vertically above the first, and at a distance h feet from it, a force **W** lbs.' weight must be exerted through h feet. Hence the work done is $\mathbf{W}h$ foot-pounds.

Next, suppose that the body is raised from a point A to a point B which is not *vertically* above A, but is at a higher level than A. Suppose that the perpendicular distance from B on the horizontal plane through A is h feet. (In this case also B is said to be h feet above A.)

The motion may be accomplished by moving the body horizontally to a position vertically below B, and next from this position to B. In the first of these motions no work is done; and in the second work, $\mathbf{W}h$ foot-pounds, is done.

Hence the work done in moving the body from A to B is $\mathbf{W}h$ foot-pounds.

Or we may show the same thing by saying: Since h feet is the resolved part of the displacement in the direction of the force **W** lbs. weight, therefore the work done is $\mathbf{W}h$ foot-pounds.

Again we have frequently to consider the work done *by* the action of gravity. Whenever a body descends vertically, or moves to a place at a lower level, the force which is the weight of the body does a quantity of work equal to the product of the weight and the depth through which the body descends vertically.

This work may be utilized in many ways. For instance on several mountain railways the weight of a stream of water is made to do useful work. Two cars are connected by a chain or rope which passes over a pulley at the top of the line. Each car has a water-tank attached to it. The one at the top has its tank filled, and by its greater weight it descends drawing up the other, the tank of which is empty. When the full car gets to the bottom of the line it is emptied, and the other, now at the top, is filled.

Energy of Fall of Water.—Again, when a mill-wheel is turned by means of a water-fall it is the weight of the water descending through a distance vertically which does the work or supplies the energy that is obtained. The gross amount of work done by the water-fall in any given time is, of course, measured by the weight of water that falls in that time and the vertical distance through which it falls. To calculate the power of the fall we must know what quantity of water falls in a given time, or the rate at which it falls.

Work done in Stretching a Rod or Wire.—Suppose that a force is applied to a rod or wire and stretches it by an amount proportional to the stretching force.

Let the elongation produced be l, and the value of the force required to produce this elongation $\mathbf{F}$.

The mean value of the force reckoned over the whole elongation is $\mathbf{F}/2$. And the entire quantity of work done is the same as that which would be done by the force $\mathbf{F}/2$ acting through the distance l.

$\therefore$ The work done is $\frac{1}{2}$ $\mathbf{F}l$, or, the work is *one half of the product of the stretching force and the elongation produced.*

Exercises VIII. a.

1. What work is done by a man in raising 120 lbs. through a vertical height of 60 feet; and if he works at $\frac{1}{10}$ H.P., how long does he take to do it?

2. How much work does a man weighing 10 stone do in ascending 30 feet; and if he does it in 1 minute, at what H.P. does he work?

3. What work does a man do in riding a bicycle 10 miles, the resistance to motion being 3 lbs.' wt.?

4. What work does a man weighing 150 lbs. do in ascending a mountain 1500 feet high?

5. If a gymnast who weighs 150 lbs. climbs a rope at the rate of 15 inches per second, show that he works at just over $\frac{1}{3}$ of a horse-power?

6. A hammer strikes a nail and drives it $\frac{1}{4}$ inch into a piece of wood. The resistance to penetration is 120 lbs.' wt., and the impact lasts for $\frac{1}{14}$ second. At what rate is work done against this resistance?

7. What work is done against gravity in pumping 1000 gallons of water to a height of 40 feet, a gallon of water weighing 10 lbs.?

8. Find the entire number of foot-tons of work done by a porter in carrying six loads of 60 lbs. each to a vertical height of 30 feet, the man himself weighing 12 stone.

9. Show that an engine working at 5 H.P. does about 4420 foot-tons per hour.

10. Find, to the nearest second, how long an engine of 22 H.P. would take to pump 10,000 cubic feet of water to a height of 45 feet, $\frac{3}{4}$ of the energy developed being utilized.

11. What must be the nett H.P. of engines and pumps required to keep a pit clear of water, if the water flows in at the rate of 4000 cubic feet per minute, and has to be pumped to an average height of 60 feet?

12. What is the ratio of the nett H.P. of an engine and crane to the indicated H.P. of the engine, when the indicated H.P. is 15, and 15 cwts. are raised by the crane through 4 feet in every second?

13. In raising 5 cwts. of stone from a quarry every 3 hours through a height of 33 feet, at what H.P. on the average is work done?

14. A body weighing 540 lbs. is drawn up by means of a rope coiled round a cylindrical drum, the diameter of which is 10 inches. How much work is done in 16 turns of the drum?

15. A cylindrical drum, 2 feet in diameter, is used to draw up a mass of 6 cwts. by means of a rope passing round the drum. If energy is developed at the rate of $7\frac{1}{2}$ H.P. at the drum, how many times does it revolve in a minute?

16. The metre being 39·370 inches and the kilogram 2·2046 pounds, show that a kilogram-metre is equal to 7·233 foot-pounds.

17. With the data of the last question show that a cheval-vapeur is about 542·5 foot-pounds per second.

18. The section of the stream which turns a mill-wheel is 10 feet × 2 feet; the water flows at 5 feet per second, and has a fall of 22 feet in turning the wheel. If the wheel gives out $\frac{4}{5}$ of the energy of the fall of the water, find its rate of output in horse-power.

19. Express a horse-power in foot-tons per hour.

20. Express a horse-power in kilogram-metres per second.

21. Why cannot a horse-power be expressed in kilogram-metres?

22. Can a horse-power be expressed in kilogram-metres per minute?

23. Criticise the statement: "A horse-power is equal to the work done in raising 550 pounds through a foot in one second."

24. A spiral spring requires a force of one pound weight to stretch it an inch. How much work is done in stretching it 3 inches? (Camb. Sr. Loc., Stat. Dyn. and Hydro., 1896.)

Work done when several Bodies are raised into new positions.—If several bodies are raised from given positions into new positions (not necessarily vertically above their old ones), it is clear that the entire work done is the sum of the products got by multiplying the weight of each body and the vertical height through which it is raised.

We shall now prove an important and useful expression for the work done in this case.

Let the weights of the various bodies be $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, etc.

Let their heights above a fixed horizontal plane be at first h_1, h_2, h_3, etc., and afterwards h'_1, h'_2, h'_3, etc.

Let the weight of all the bodies together be $\mathbf{W}$. And let the heights of their common Centre of Gravity above the plane, before and after moving, be H and H'.

The work done is

$$\mathbf{w}_1(h_1'-h_1)+\mathbf{w}_2(h_2'-h_2)+\mathbf{w}_3(h_3'-h_3)+\text{etc.}$$

$$=\mathbf{w}_1h_1'+\mathbf{w}_2h_2'+\mathbf{w}_3h_3'+\ldots-(\mathbf{w}_1h_1+\mathbf{w}_2h_2+\mathbf{w}_3h_3+\ldots)$$

$$=\mathbf{W}H'-\mathbf{W}H=\mathbf{W}(H'-H).$$

Thus the entire work done is *the product of the weight of all the bodies moved, and the height through which the common centre of gravity is raised.*

In what has just been done we may suppose that each body is a very small particle, so that the whole of it is practically situated at one point, the distance through which it is raised being thus the vertical distance between its two point positions; or else we may suppose that each body is raised without rotation, so that each point of a body is raised through the same distance as any other point of the same body.

Suppose a body of extended dimensions to be moved from one position to another. To find the entire work done against gravity. This is the same as the work done in moving the various elementary particles into their new positions. And, as we have just seen, this is equal to the sum of the weights of the particles × height through which their C.G. is raised; that is, it is equal to

Weight of body × height through which its C.G. is raised.

EXAMPLE.—Find the work done against gravity in drawing water from a depth of 12 feet below the surface of the earth and filling a rectangular tank 3 feet deep × 6 feet × 4 feet, the bottom of the tank being 10 feet above the earth.

Volume of water raised $= 4 \times 6 \times 3$ cub. ft.

$\therefore$ Mass of water raised $= 4 \times 6 \times 3 \times 62\frac{1}{2}$ lbs.

Centre of gravity of water is raised through

$12 + 10 + 1\frac{1}{2}$ feet $= 23\frac{1}{2}$ feet.

$\therefore$ Work done $= 4 \times 6 \times 3 \times 62\frac{1}{2} \times 23\frac{1}{2}$ ft. lbs.

$= \mathbf{105750}$ **ft.-lbs.**

EXAMPLE.—A rectangular block of wood measuring 6 inches × 8 inches × 10 inches, and weighing 27 lbs. per cubic foot, stands on one of its 6 × 10 faces. Find the work done in pushing it over, so that it falls on one of its 8 × 10 faces.

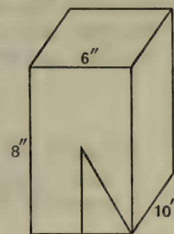

FIG. 106.

In pushing the block over work is done till its C.G. comes just above the edge about which it turns. After this the block falls over.

At that instant the C.G. is 5 inches above the plane, and it was 4 inches above, so that it is made to rise 1 inch.

$$\text{Mass of block} = \frac{6 \times 8 \times 10}{1728} \times 27 \text{ lbs.} = \frac{15}{2} \text{ lbs.}$$

$\therefore$ Work done $= \frac{1}{12} \times \frac{15}{2}$ ft.-lbs.

$= \frac{5}{8}$ **ft.-lbs.**

Indicated and Effective Power.—By the power of an engine may be meant

(*a*) The rate at which work is done by the pressure of the steam or gas as it expands; or

(*b*) The rate at which useful work is got from the engine. This is less than (*a*) by the rate at which work is lost in overcoming frictional resistances, etc., in the parts of the engine.

(*a*) may be called the *gross* and (*b*) the *nett* power of the engine.

Expressed in horse-power they are generally spoken of as the *indicated* and the *effective* H.P. respectively.

Work done by the pressure of Steam in an Engine.—Consider the piston of the engine moving to and fro in its cylinder, under a pressure of steam on one side only as it moves one way, and on the other side as it moves the other way.

Let the area of the face of the piston be A square inches.

Let the length of the stroke, that is the single stroke, measured one way only, or the distance between the extreme positions of the piston, be l feet.

Let the pressure of steam acting on the piston be $\mathbf{p}$ lbs.' wt. per square inch.

Let the number of single strokes made in a minute be n.

Then the entire pressure acting on the piston is $\mathbf{p}A$ lbs.' wt.

The work done in one stroke is $\mathbf{p}Al$ foot-pounds.

The work done per minute is $\mathbf{p}Aln$ foot-pounds.

The H.P. at which steam pressure works is

$$\frac{\mathbf{p}Aln}{33,000}.$$

EXAMPLE.—A steamer is going at the rate of 16 miles an hour; the H.P. of her engines is 5000. What is the resistance to motion through the water?

[In this question we require, of course, the effective H.P. of the engines; and this is clearly what is meant to be given as 5000.]

Let the required resistance be $\mathbf{F}$ lbs.' weight.

The distance travelled in one minute is

$$\frac{16 \times 5280}{60} = 16 \times 88 \text{ feet.}$$

$\therefore$ the work done in 1 min. $= 16 \times 88\,\mathbf{F}$ ft.-lbs.

Also the work done in 1 min. $= 5000 \times 33000$ ft.-lbs.

$$\therefore \quad 16 \times 88\,\mathbf{F} = 5000 \times 33000,$$

$$\mathbf{F} = \frac{5000 \times 33000}{16 \times 88}$$

$$= 117,000 \text{ about.}$$

The resistance is **about 117,000 lbs.' weight.**

Questions on Bicycles.—To say that a bicycle is geared to so many inches means that it travels as far for one complete revolution of the cranks (or from the instant when either foot of the rider is in its lowest position to the next instant when it is again in its lowest position) as a wheel, whose diameter is the same number of inches, would travel in one revolution when rolling along the ground.

EXAMPLE.—A bicycle is geared to 64 inches. The rider pedals at the rate of 70 revolutions per minute, and works at $\frac{1}{16}$ H.P. What is the resistance to motion?

Let the resistance to motion be **F** lbs.' wt.

The distance travelled in 1 minute is

$$\tfrac{22}{7} \times \tfrac{64}{12} \times 70 \text{ feet.}$$

$\therefore$ work done in 1 minute

$$= \tfrac{22}{7} \times \tfrac{64}{12} \times 70 \times \mathbf{F} \text{ ft.-lbs.}$$

$$\therefore \tfrac{22}{7} \times \tfrac{64}{12} \times 70 \times \mathbf{F} = \tfrac{1}{16} \times 33000.$$

$$\mathbf{F} = \frac{33000 \times 7 \times 12}{16 \times 22 \times 64 \times 70}.$$

Required resistance = **1·8 lbs.' weight, about.**

Exercises VIII. b.

1. In rolling a cylinder, 2 feet in diameter, and weighing 2 cwts., along a horizontal road it comes to a ridge across the road $1\frac{1}{2}$ inches high. How much work is done in getting the cylinder past the ridge?

2. What is the work done against the action of gravity in bringing the earth from a well, 4 feet wide and 10 feet deep, to the level of the surface, the earth weighing 120 lbs. per cubic foot?

3. What work is done in rolling up a blind on a roller at the top, the blind being 6 feet long and weighing 3 pounds?

4. What work is done in drawing a bucket of water from a well, 40 feet deep, the bucket weighing 8 lbs., the water 100, and the rope 9 oz. per foot?

5. The diameter of the piston of a steam engine is 10 inches; the length of the stroke is 3 feet. The engine is working with a pressure of 300 lbs.' wt. to the square inch on the piston, and making 90 strokes per minute. What H.P. does it develop?

6. An engine has two pistons each of 30 centimetres diameter; the length of the stroke being 1·2 metres. The fly-wheel, to which the pistons are directly connected, makes 40 complete revolutions per minute; and in each stroke the mean excess of the pressure on one side of a piston over that on the other is 40 kilograms' wt. per square centimetre. Find the rate at which the engine is working in chevals.

7. A cylindrical pit, 20 feet in diameter and 40 feet deep, is filled to a depth of 30 feet with water. Show that a 50 H.P. engine will empty it in about 9 minutes.

8. Find the number of foot-tons of work done in raising 16 cwts. of stone through 40 feet by means of a chain weighing 12 pounds per foot.

9. A cylinder, 7 feet long and having diameter of end 2 feet and weighing 800 lbs., lies on a horizontal plane. How much work must be done to raise it up on its end?

10. A book, 8 inches long, 1 inch thick, and weighing 2 pounds, stands up on end. What work must be done to push it over?

11. A man does 1,027,200 foot-pounds of work in 8 hours; what is his power, the units being foot-pounds and minutes? (Science and Art, 1897.)

12. A machine is contrived, by means of which a weight of 3 tons, by falling 3 feet, is able to lift a weight of 168 lbs. to a height of 100 feet. Find the work done by the falling body, and what part of the work is used up in overcoming the friction of the machine. (Science and Art, 1897.)

13. Define a foot-pound and a horse-power.

A steam crane raises a weight of 5 tons uniformly through a height of 110 feet in 40 seconds. Find at what H.P. it is working. (Science and Art, 1898.)

14. The base of a cylinder has a diameter of 3 feet, and its height is 4 feet; the cylinder is of uniform density and weighs 25 cwts. Find how many foot-pounds of work must be done in throwing it over. (Science and Art, 1898.)

Summary.

Work is done by an agent when this exerts a force through a distance.

The measure of the work done is

(*a*) The product of the displacement and the resolved part of the force in the direction of the displacement; or,

(*b*) The product of the force and the resolved part of the displacement in the direction of the force;

and it may be positive, zero, or negative.

The **unit of work** is that done when unit force acts through unit distance.

The **foot-pound**, or British Engineers' unit of work, is the work done when the force of a pound's wt. acts through a foot.

The **kilogram-metre** is the work done when a force equal to the weight of a kilogram acts through a metre.

Power means rate of doing work, and may be expressed in foot-pounds per second.

A **horse-power** is the rate of doing work at 33,000 foot-pounds per minute, or 550 foot-pounds per second.

A **cheval-vapeur** is the rate of doing work at 75 kilogram-metres per second. It is equal to about 542 foot-pounds per second; and is thus a little less than a horse-power.

Energy means the same thing as work, and is generally used as applying to the work which can be got from an agent in given circumstances.

The work done *against* gravity in raising a body is equal to the weight of the body × the height through which it is raised vertically.

The work done *by* the action of gravity when a heavy body descends is equal to the weight of the body × the vertical distance through which it descends.

The work done against gravity in raising any bodies to new positions is equal to the product of the weight of all the bodies and the vertical height through which the centre of gravity of the bodies is raised.

The **gross** or **indicated** H.P. of an engine may be calculated from the data of its dimensions, rate of moving, and steam-pressure.

The **nett** or **effective** H.P. is less than this because of the work lost in friction, etc.

CHAPTER IX.

MACHINES. MECHANICAL ADVANTAGE. EFFICIENCY. LEVERS AND INCLINED PLANE.

Machines.—A Machine is an arrangement for transmitting a force from one line of action to another.

In the process of transmission by a machine the magnitude of the force is, in general, altered; it may be increased or diminished.

In Mechanics there are five simple machines, sometimes called the Mechanical Powers. These are (1) The Lever, (2) The Inclined Plane, (3) The Pulley, (4) The Wheel and Axle, (5) The Screw.

In investigating the properties of any one of these machines we shall consider the relation between the magnitude of a force applied to the machine and that of the force transmitted by the machine, or, in other words, the relation between the force applied and the resistance which the force overcomes by the help of the machine.

In these cases the force applied to a machine is frequently spoken of as a *power*. But it should be noticed that this is not a very accurate use of this word, the proper meaning of the word *power* in Mechanics being rate of doing work.

The force applied has also been called the **Effort.**

In finding the relation between the forces we shall, in general, suppose that the parts of the machine are frictionless. If there is friction this relation will be modified. When the force applied acts to overcome the resistance and set the machine in motion against it, the friction acts to oppose the force, and tends to keep the machine at rest. Hence, in consequence of friction in a machine the force necessary to overcome a given resistance is always greater than it would otherwise be.

Mechanical Advantage.—In any machine the ratio of the resistance overcome to the force applied, or the effort, is called the *mechanical advantage.* This ratio may be a quantity greater or less than unity.

In a case in which there is no friction the work done by the force is exactly equal to the work done against the resistance or the effective work produced. But, in consequence of friction, some of the work expended is always lost, going to produce heat in the parts of the machine where friction acts. Hence the work obtained is less than that expended.

Efficiency of a Machine.—In any machine the ratio of the useful work to the work expended is called the *efficiency.*

It is clear that by diminishing friction we increase the efficiency, and the ultimate ideal value of the efficiency, when all friction has been eliminated, is unity. This is a value which is never completely attained in any machine. And the more complicated a machine the more friction must there be in its parts, and the less is the efficiency which we can expect from it.

The efficiency of a machine is not a constant quantity. It will depend on the care taken of the machine, that is, on the lubrication of its parts, etc., perhaps also on the speed at which it is working, and on the forces in action, the efficiency for one resistance being different from that for another.

In any case the meaning of efficiency is the ratio

$$\frac{\textbf{useful work obtained}}{\textbf{work expended}}$$

in the given circumstances.

Other Expressions for Efficiency.—Two other useful expressions for the efficiency, in terms of forces, can be obtained.

Suppose that with a given machine, **P** and **R** are corresponding values of the force applied and the resistance overcome on the supposition that no work is lost, that is that the efficiency is unity.

Let a and b be corresponding values of the distance through which **P** acts and the distance through which **R** is overcome.

Then, since the works done by **P** and against **R** are equal,

$$\mathbf{P}a = \mathbf{R}b.$$

Now let **P′** be the actual force required to overcome the resistance **R** and to work the machine.

Then the work expended is **P′**a.

And work obtained is **R**b.

$$\therefore \text{ efficiency} = \frac{\mathbf{R}b}{\mathbf{P}'a} = \frac{\mathbf{P}a}{\mathbf{P}'a} = \frac{\mathbf{P}}{\mathbf{P}'}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Again, let **R′** be the resistance which the force **P′** is able to overcome in working the machine.

Then work expended is **P**a.

And work obtained is **R′**b.

$$\therefore \text{ efficiency} = \frac{\mathbf{R}'b}{\mathbf{P}a} = \frac{\mathbf{R}'b}{\mathbf{R}b} = \frac{\mathbf{R}'}{\mathbf{R}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

These expressions (1) and (2) for the efficiency are useful, and they are easily remembered.

(1) is ratio of theoretical to actual applied force ;

(2) is ratio of actual to theoretical resistance overcome.

Each is less than unity, as an efficiency always is, for the theoretical force applied for a given resistance is less than the actual ; and the actual resistance overcome by a given force is less than the theoretical.

Velocity Ratio.—Suppose that in any machine while the applied force **P** acts through a distance a, the resistance **R** is overcome through a distance b.

If the machine is one in which the motion of the point of application of **P** always bears the same ratio to the motion of the point of application of **R**, we may suppose a and b to be any corresponding displacements of any magnitude ; but if, as for the lever, this is not the case, so that the ratio of a to b depends on the amount of the displacement of **P** and on the position of the machine to begin with, we must suppose here that a and b are both extremely small magnitudes.

Then the ratio $\frac{a}{b}$ is called the **velocity ratio** of the machine.

In the lever, in which a is not always proportional to b for large displacements, as we shall see, the velocity ratio will depend on the inclinations of the forces to the bar.

In an ideal machine, in which no work is lost in friction, the works done by **P** and against **R** are equal, that is

$$\mathbf{P}a = \mathbf{R}b.$$

Thus the mechanical advantage, which is $\frac{\mathbf{R}}{\mathbf{P}}$, is equal to $\frac{a}{b}$.

Or, in the ideal case,

mechanical advantage = velocity ratio.

In practice the mechanical advantage is always less than the velocity ratio.

Let E be the efficiency of the machine.

$$E = \frac{\text{useful work obtained}}{\text{work expended}} = \frac{\mathbf{R}b}{\mathbf{P}a} = \frac{\mathbf{R}}{\mathbf{P}} \div \frac{a}{b}.$$

Efficiency = mechanical advantage ÷ velocity ratio;

or, *Mechanical advantage = velocity ratio × efficiency.*

Levers.—A Lever is a rigid bar which can be turned about a fixed point acting as a support on which some point of the lever rests. The fixed supporting point is called a **fulcrum.**

The bar or rod of which a lever consists is generally straight, but it may be bent, and so form a *bent lever.*

Let two forces, **P** and **R**, act on the lever ACB having a fulcrum at C.

P may be taken as the force applied or effort, and **R** as the resistance to be overcome, considered as another force acting *on* the lever.

Suppose the weight of the lever to be negligible.

The condition for equilibrium between **P** and **R** is that the moments of **P** and **R** about C shall be equal and in opposite senses.

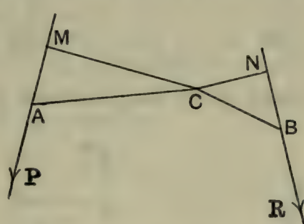

Fig. 107.—Forces acting on lever and producing equilibrium.

Thus, if CM, CN are perpendicular to the lines of action of **P** and **Q**,

$$\mathbf{P} \cdot CM = \mathbf{Q} \cdot CN.$$

The points A, B, C (that is, the points of application of the force and resistance, and the fulcrum) may be arranged in a different order, as the next two figures show. But the condition given above is always the necessary and sufficient condition of equilibrium for the lever.

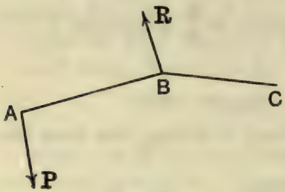

FIG. 108.

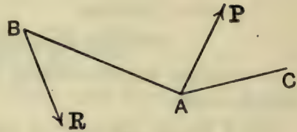

FIG. 109.

In general, unless the contrary is specified, a lever is supposed to be straight and of negligible weight, and the forces are supposed to act at right angles to it.

Classes of Levers.—Levers are divided into three classes according to the relative positions of the fulcrum and the points of application of the resistance and the force applied. The class is called first, second, or third according as the Fulcrum, Resistance, or Effort (also called Power) is in the middle. This will be easily remembered by remembering the letters **F**, **R**, **P** in the order here given.

The figures here given show the three classes with the forces

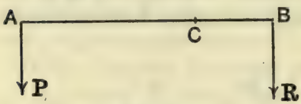

FIG. 110.

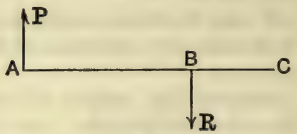

FIG. 111.

acting, in the simplest cases, in which the levers are straight, and the forces at right angles to them.

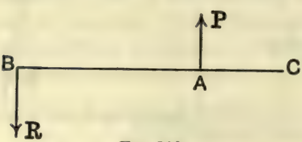

FIG. 112.

In any case we have for equilibrium,

$$\mathbf{P} \cdot CA = \mathbf{R} \cdot CB.$$

$$\therefore \frac{\mathbf{R}}{\mathbf{P}} = \frac{CA}{CB}.$$

That is, the Mechanical Advantage is equal to the fraction $\frac{CA}{CB}$.

In the first case C may be anywhere between A and B, so that CA and CB may have any relative values.

In the second case CA must be greater than CB.

In the third case CA must be less than CB.

Hence, in a lever of the first class, the mechanical advantage may have any value ; in a lever of the second class it is greater than unity, and in a lever of the third class it is less than unity.

Examples of the classes of Levers.—*Class I.*—A crow-bar used in the ordinary way to raise a stone or other heavy body at one end, pressing down on some support used as a fulcrum near this end, the hands supplying the effort by pressing down at the other end.

A poker used to raise coals in a grate.

Class II.—An oar, the resistance to be overcome being at the rowlock by which the boat is driven forward, the water against which the blade of the oar pushes acting as fulcrum.

A wheel-barrow.

Class III.—The limbs of animals ; for example, the fore-arm used to hold up a weight ; the elbow joint is the fulcrum, the effort is applied by the biceps muscle acting at a point in front of the elbow, and the resistance is overcome at the hand.

Examples of double levers of the three classes are to be found respectively in (1) pliers, (2) nut-crackers, (3) sugar-tongs.

Principle of Work applied to Levers and to Machines in general.—In the case of any of the simple machines which we consider we may show, from the relation existing between the force applied and the resistance overcome when there is no friction, and from the geometrical relation which we may determine between the displacements of the points of application of the force and the resistance, that the work done by the force is always equal to that done against the resistance.

Or, on the other hand, by assuming this principle we may determine the relation between the force and the resistance.

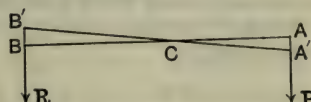

Fig. 113.

Suppose, for example, that we have a lever ABC of the first class, fulcrum at C, with forces **P** and **R** acting at A and B at right angles to the lever, and in equilibrium.

Now suppese that the lever undergoes a very small displacement to the position $A'CB'$. AA', BB' are approximately short straight lines in the directions of **P** and **R**.

The work done by **P** is **P** . AA'.

That done against **R** is **R** . BB'.

And we have the geometrical relation

$$\frac{AA'}{BB'}=\frac{CA}{CB}.$$

Now, if we assume that

$$\mathbf{P} \,.\, CA=\mathbf{R} \,.\, CB,$$

it follows that $\mathbf{P} \,.\, AA'=\mathbf{R} \,.\, BB'$.

Thus the two quantities of work are equal.

Again, if we assume that

$$\mathbf{P} \,.\, AA'=\mathbf{R} \,.\, BB',$$

it follows that $\mathbf{P} \,.\, CA=\mathbf{R} \,.\, CB$,

the relation that must exist between **P** and **R**.

If the force **P** is just sufficient to equilibrate **R**, it will not, of course, produce any displacement. But supposing the entire absence of all such resistances as friction forces, a force which is ever so little greater than **P** will displace the lever and do work against the action of **R**. The work done by this force, since it is greater than **P**, is greater than the work which would be done by **P** for the same displacement, but can be made to differ from the work of **P** by as little as we please.

It is in this sense that we may say that the work done by the applied force is **P** . AA', and that this is equal to **R** . BB'.

In another sense, too, we may say that the work done by the applied force, whatever this may be, is equal to **P** . AA' or to **R** . BB', where **P** is the force necessary to equilibrate **R**. The residue of the work done by the force goes to set the lever in motion, according to the principles of Dynamics.

Similar remarks to these apply to the work done by means of other machines.

Experiment 25.—Take a rod about 3 feet long, and fasten it by a nail or screw passing through a hole in the middle O to a vertical support, so that it turns freely about the nail in a vertical plane and balances on it.

Hang a weight, say about 4 or 5 lbs., to one end, and by means of a spring-balance connect a point of the rod about midway between this end and its middle point to a point about 10 or 12 inches vertically above *O*, so that it remains horizontal.

Determine the moment of the weight of the body suspended at the end about *O*, and, by observing the indication of the balance, and measuring the perpendicular distance from *O* to the string attaching the balance to the rod, find the moment of the pull in the balance about *O*.

These two moments should be equal.

Experiment 26.—With the same rod hand a heavy weight, about 20 lbs., or as much as the rod and nail will safely bear, close to *O*, and maintain equilibrium by pulling down with the spring-balance, keeping the rod horizontal. Measure the short distance from *O* to the weight carefully. The spring-balance may pull down either vertically or obliquely.

Determine the moments about *O* of the forces acting on the two sides of *O*, and show that these are equal.

Exercises IX. a.

1. A crow-bar 4 feet long is used to raise a stone at one end, resting on a support two inches from this end. The upward pressure against the stone that is required is 600 lbs.' wt. With what pressure must the other end of the bar be pressed down?

2. A piece of wire is held in the jaws of a pair of pliers at $\frac{3}{4}$-inch from the joint. The hands press the handles together with a force of 25 lbs.' wt. at 6 inches from the joint. What is the pressure on the wire?

3. The load in a wheel-barrow is 135 lbs., and its weight acts at 20 inches from the centre of the wheel. The horizontal distance from the axle to the point at which each handle is seized is 6 feet 6 inches. Find the upward force with which each handle must be lifted.

4. The arms of a lever are 4 inches and 4 feet, and it is found that to overcome a resistance of 120 lbs.' wt. a force of 10·3 lbs'. wt. must be exerted, some of the work done being lost in friction, etc. What is the efficiency of the lever? and what fraction of the work done is lost?

5. A pair of sugar-tongs holds a lump of sugar at a distance of $4\frac{1}{2}$ inches from the end at which the two members join. The hand presses the tongs with a force of 14 oz. wt. at a distance of $2\frac{1}{2}$ inches from this end. A force of 8 oz. wt. would be sufficient to compress the tongs as much with nothing between. What is the pressure on the lump of sugar?

6. A body weighing 20 lbs. is held up in the hand, the fore-arm being horizontal and the upper arm vertical. If the point of attachment of the biceps is 2 inches in front of the point about which the fore-arm turns, and the weight of the body acts at $12\frac{1}{2}$ inches from the same point, what is the tension in the muscle?

7. Draw an equilateral triangle ABC, and let BC represent a weightless lever acted on at B by a force of 7 units from A to B, and at C by a force of 9 units from A to C; if the lever is at rest, find, by construction or otherwise, the position of the fulcrum; find also the magnitude and direction of the pressure on the fulcrum. (Science and Art, 1897.)

The Inclined Plane.—This is a plane surface inclined at an angle to the horizon.

If the plane is quite smooth and a body is on it, the force acting up the plane and parallel to it, necessary to keep the body from sliding down, or necessary to draw the body up the plane, is less than the weight of the body. This is the most common direction for the force; but it may act in other directions. We shall consider the relations between the force and the weight of the body in this case and in the case in which the force acts horizontally.

In the figure let AB represent a line on the face of the inclined plane which is *a line of greatest slope*, that is, so drawn as to have as great an inclination as possible to the horizon. Such a line will cut at right angles the horizontal lines drawn on the face of the plane; and it will be such that a body placed on the plane at any point of this line tends to slide down the plane along the line.

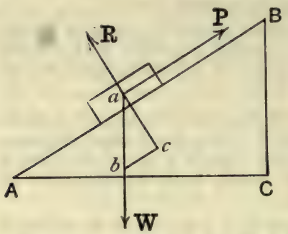

FIG. 114.—Principle of inclined plane. Effort parallel to plane.

Let AC be a horizontal line drawn vertically below AB, that is, so that AB and AC are in one vertical plane; and let BC be vertical.

AB is called the *length*, AC the *base*, and BC the *height* of the plane.

Let them be denoted, respectively, by l, b, h.

(a) Suppose a body of weight **W** on the plane is sustained by a force **P** acting parallel to AB.

A third force, the reaction of the plane, acts on the body. Call this reaction **R**. Since the plane is smooth this force is at right angles to its surface.

Let a be the point in which **W**, **P**, **R** meet.

Draw bc parallel to AB to form, with the lines of action of **W** and **R**, the triangle abc.

Then, $\because$ ac and ab are at right angles to AB and AC,

$$\therefore \angle bac = \angle BAC.$$

And $\angle^s$ at c and C are equal.

$$\therefore \triangle^s\ abc,\ ABC \text{ are similar.}$$

But abc is triangle of forces for **P**, **W**, **R**.

$$\therefore \frac{\mathbf{P}}{bc} = \frac{\mathbf{W}}{ab} = \frac{\mathbf{R}}{ca}.$$

$$\therefore \frac{\mathbf{P}}{BC} = \frac{\mathbf{W}}{AB} = \frac{\mathbf{R}}{CA}.$$

$$i.e. \quad \frac{\mathbf{P}}{h} = \frac{\mathbf{W}}{l} = \frac{\mathbf{R}}{b}.$$

(b) Suppose a body of weight **W** is sustained by a force **P** acting horizontally.

Draw bc horizontally to form, with the lines of action of **W** and **R**, the triangle abc.

Then $\triangle\ abc$ is similar to $\triangle\ ABC$.

$$\text{And} \quad \frac{\mathbf{P}}{cb} = \frac{\mathbf{W}}{ac} = \frac{\mathbf{R}}{ba}.$$

$$\therefore \frac{\mathbf{P}}{CB} = \frac{\mathbf{W}}{AC} = \frac{\mathbf{R}}{BA}.$$

$$i.e. \quad \frac{\mathbf{P}}{h} = \frac{\mathbf{W}}{b} = \frac{\mathbf{R}}{l}.$$

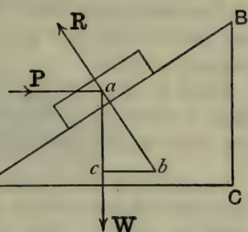

FIG. 115.—Principle of inclined plane. Effort horizontal.

The mechanical advantages, *i.e.* the values of **W**/**P**, are in the two cases $\frac{l}{h}$ and $\frac{b}{h}$.

The most important thing to bear in mind is the ratio of **P** to **W** in case (a). This it is not difficult to remember. In an ordinary inclined plane h is small as compared with b and l. In this case only a small force is required to sustain the weight. **P** is proportional to the height h. **W**, the weight which **P** can sustain, is proportional to the greater of the remaining two sides. We may notice that here **P** acts at a great advantage ;

the mechanical advantage is the greatest ratio that can be formed of the three sides, h, l, b.

Principle of Work applied to Inclined Plane.—As an example of the application of the principle of work let us consider the relation between **P** and **W** in case (*a*).

Here, there is a third force **R** acting on the body. But the displacement of the body is always in a direction at right angles to **R**, so that **R** does no work.

Hence if **P** is the force necessary to counterbalance **W**, the work done by **P** in displacing the body by any distance up the plane is equal to the work done against **W**.

Now let us suppose the body drawn up the whole length l of the plane, and consequently raised through a vertical height h.

The work done by **P** is $\mathbf{P} \,.\, l$.

That done against **W** is $\mathbf{W} \,.\, h$.

$$\therefore \quad \mathbf{P}l = \mathbf{W}h.$$

Inclination of Plane.—The inclination of an inclined plane is frequently specified by mentioning the vertical height that the plane rises in a given length, measured along its face or sloping surface.

Thus if the slope of a plane is said to be 1 in 5, it is meant that it rises by 1 unit for every 5 units *along its face;* or that the ratio *height* : *length of sloping surface* = 1 : 5.

EXPERIMENT 27.—Take a flat, smooth board about 2 or 3 feet long and 8 inches wide or upwards. Rest one end on a horizontal table and raise the other end about $\frac{1}{4}$ or $\frac{1}{3}$ of the length of the board.

Get a small trolly or carriage small enough to run on the board and with well-made, smoothly running wheels. Load it to bring its entire mass with load up to about 8 or 10 pounds.

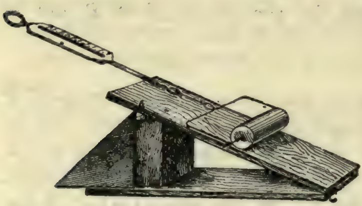

FIG. 116.—Experiment with inclined plane. Effort parallel to plane.

Instead of the trolly a turned cylinder of heavy wood with two small screws or nails at its ends carried by a wire as shown in the figure may be used.

Use a spring balance to hold the trolly or cylinder on the slope, the balance pulling directly up the plane and parallel to its surface, and observe the pull shown by the balance.

Find the ratio of the pull of the balance to the weight of the trolly or cylinder, and by measurement find the ratio of the height to the length of the inclined plane formed by the board.

These two should be approximately equal.

On account of the friction of the trolly the spring balance will probably not give a perfectly definite indication. Take the reading when the trolly is on the point of being pulled up and that when it is on the point of going down; and use for the pull the mean of these two.

EXPERIMENT 28 (Modification of last experiment).—Use the same board and trolly or cylinder. Fix a smooth pulley to the top of the board. Pass a string from the trolly over the pulley and attach such weights to its other end as will sustain the trolly on the board. The best value to take is the mean between the extreme weights that will maintain equilibrium.

Compare, as before, the ratios of pull to weight and of height to length of plane.

EXPERIMENT 29.—Using the same board and trolly or cylinder attach a spring balance to the weight to sustain it on the plane by means of a horizontal pull as shown in the figure.

Determine the mean value of the pull as before.

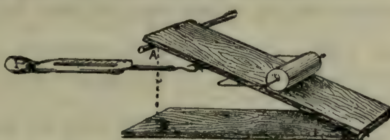

FIG. 117.—Experiment with inclined plane. Effort horizontal.

Find the ratio of pull to weight; and find by measurement the ratio of height to length of plane.

These two should be approximately equal.

EXAMPLE.—An engine working at H horse-power draws a train of mass W tons up an incline of 1 in s, at V miles per hour. Show that the frictional resistance to motion is $\frac{375H}{V} - \frac{2240\,W}{s}$ lbs.' wt.

Let the required resistance be **F** lbs.' weight.

The force necessary to overcome the weight of the train on the incline is

$$\frac{W}{s} \text{ tons' weight} = \frac{2240\,W}{s} \text{ lbs.' weight.}$$

V miles are passed over in 1 hour,

i.e. $V \times 5280$ feet are passed over in 3600 secs.

$\therefore \frac{22V}{15}$ feet are passed over in 1 sec.

$$\therefore \text{ work done in 1 second by engine} = \frac{22V}{15}\left(\frac{2240W}{s} + \mathbf{F}\right).$$

$$\therefore \frac{22V}{15}\left(\frac{2240W}{s} + \mathbf{F}\right) = 550H.$$

$$\frac{2240W}{s} + \mathbf{F} = \frac{15 \times 550H}{22V}.$$

$$\mathbf{F} = \frac{375H}{V} - \frac{2240W}{s}.$$

Exercises IX. b.

1. The height of an inclined plane is to its length as 3 is to 5. Find the force acting parallel to the face of the plane necessary to sustain a mass of 100 lbs. on the plane, and find the pressure on the plane.

2. With the same plane find the force acting horizontally which will hold the mass of 100 lbs. on the plane, and find the pressure on the plane.

3. What work is done by a horse against the action of gravity in drawing a carriage with its load, all weighing 1000 lbs., 100 yards up a slope of 1 in 25?

4. To draw 100 lbs. up a plane 40 yards long the force required is 8 lbs.' wt. What is the height of the plane?

5. The work done in drawing 100 lbs. along 24 feet of an inclined plane is 300 foot-pounds. Show that the slope of the plane is 1 in 8.

6. The base of an inclined plane is 17 feet and the height 2 feet. Find the work done in drawing a body weighing 100 lbs. up 17 feet of the slope.

7. An inclined plane is 4 feet high; and to sustain a body on it a force equal to $\frac{1}{3}$ of the weight of the body, acting parallel to the face of the plane, is required. What is the length of the base of the plane?

8. How many foot-tons of work are done by an engine against the action of gravity in drawing a train of 60 trucks up one mile of an incline of 1 in 50?

9. A body is held on a smooth inclined plane of slope 1 in 50 by means of a force acting parallel to the face of the plane. Show that the pressure on the plane is less than the weight of the body by about $\frac{1}{5000}$ of the weight.

10. A body rests on a rough plane. The plane is tilted up till it makes an angle of 30° with the horizon, and the body then begins to slide; show that the friction force at that instant is half the weight of the body.

11. An engine pulls a train of mass W tons down an incline of 1 in s, at the rate of V miles an hour. The frictional resistance is w lbs.' wt. per ton. Show that the H.P. of the engine is $\frac{VW}{375}\left(w - \frac{2240}{s}\right)$.

12. A cyclist and his machine weigh 180 lbs. To ascend a slope of 1 in 30 at 5 miles an hour, at what H.P. must he work (1) if there is no frictional resistance, (2) if the frictional resistance is 3 lbs.' wt.?

13. What must be the inclination of the plane that a force P will support the same weight whether it acts horizontally or parallel to the length of the plane? (Coll. Precep., 1897.)

14. When a force F, acting horizontally, supports a weight W on an inclined plane rising a height of 28 to a length of 100, find the ratio of F to W. (Coll. Precep., 1898.)

15. A body weighing 150 kilograms rests on a smooth inclined plane which rises 2 cm. in a horizontal distance of one metre. What distance along the plane would the expenditure of one million ergs enable one to move the body? N.B.—Gravity may be taken as 981 dynes per gram. (London Matric., 1898.)

16. A lever, 2 feet long, has a power equal to the weight of 10 pounds acting at one end, 18 inches from the fulcrum. What is the greatest weight it will support at its other end? (Oxford Locals, 1899.)

17. A vertical force f is applied to a horizontal straight uniform lever of length l at a distance d from one end of it. The weight of the lever is W. Find the distance x from the same end of the lever at which a fulcrum must be placed in order that the lever may be in equilibrium. (London Matric., 1899.)

18. Find what resistance acting vertically downwards at the end of a uniform poker, which rests horizontally on the bar of a grate with 3 inches projecting into the fire, can be overcome by a downward force of 10 lbs.' wt. applied at the other end. The poker is 30 inches long and weighs 4 lbs. (Camb. Jr. Loc., Mech., 1897.)

19. A body weighing $4\sqrt{3}$ lbs. rests on a smooth inclined plane whose length is twice its height, under the action of a horizontal force of 2 lbs.' wt. directed towards the plane, the reaction of the plane 7 lbs.' wt., and another force; determine the magnitude and direction of the last. (Camb. Jr. Loc., March, 1897.)

20. What horse-power is required to draw a weight of 1 ton up a smooth plane, inclined at 30°, at the rate of 20 feet per minute? (Science and Art, 1898.)

21. Find in magnitude and direction the least force which will keep a body weighing 100 lbs. at rest on a smooth inclined plane, inclined at an angle of 45° to the horizontal. (*N.B.*—A graphical solution will be accepted.) (Science and Art, 1898.)

Summary.

A **machine** is an arrangement for transmitting force. The magnitude of the force is generally altered in the process.

The force applied to a machine is called the **Effort** (sometimes called Power, but this is not a good term for it); the force overcome is called the **Resistance**.

$$\textbf{Mechanical advantage} = \frac{\textbf{Resistance}}{\textbf{Effort}}.$$

In any practical case the work done by the effort exceeds that done against the resistance, because some work is lost in overcoming friction, etc., in the parts of the machine.

In a machine

$$\textbf{Efficiency} = \frac{\textbf{Work obtained}}{\textbf{Work expended}}.$$

If a and b are infinitesimally small distances through which the effort acts, and the resistance is overcome at the same time in any machine, then

$$\textbf{Velocity ratio} = \frac{a}{b}.$$

$$\textbf{Mechanical advantage} = \textbf{Velocity ratio} \times \textbf{Efficiency}.$$

Levers.—The general necessary and sufficient condition of equilibrium is that the moments about the fulcrum of the effort and the resistance should be equal and opposite.

Levers are divided into three classes according as *Fulcrum*, *Resistance*, or *Effort* is in the middle.

In any simple machine, if there is no friction, the work done by the effort may be shown to be equal to that done against the resistance. Or, assuming this principle, we can find the relation between the effort and the resistance.

With a **smooth inclined plane** suppose **P** is the effort necessary to sustain a body of weight **W** on the face of the plane, and **R** is the normal resistance of the plane. Let the height, length, and base be h, l, b.

(a) If **P** is parallel to the face of the plane,

$$\frac{\mathbf{P}}{h} = \frac{\mathbf{W}}{l} = \frac{\mathbf{R}}{b}.$$

(b) If **P** is horizontal,

$$\frac{\mathbf{P}}{h} = \frac{\mathbf{W}}{b} = \frac{\mathbf{R}}{l}.$$

CHAPTER X.

PULLEYS. WHEEL AND AXLE. SCREW. TOOTHED WHEEL.

Pulley.—A pulley is a wheel which can rotate on an axis passing through its centre, and having a groove round its circumference in which a string can lie.

If the axis of the pulley is fixed, so that it is capable of no motion but mere rotation, it is called a **fixed pulley.**

If the axis is fixed in a frame or block, which can be easily moved, the pulley is called a **movable pulley.**

We shall suppose, in general, that the axis of a pulley is quite smooth, and that the string passing round it is perfectly flexible, so that no effort is required to bend or unbend it. In these circumstances the tensions in the two parts of a string on the two sides of the piece which lies in the groove of a pulley, forming part of a system in equilibrium, are equal to one another.

A fixed pulley is used merely to alter the direction of the pull produced by a string; by its use we alter neither the magnitude of the force applied nor the distance through which displacement is produced. Considered as a machine, its mechanical advantage is unity. Or, as it is sometimes said, no mechanical advantage is gained by using it.

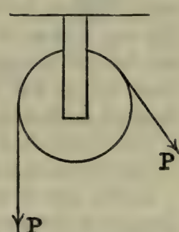

FIG. 118.—Fixed pulley.

Single movable Pulley.—Suppose a single movable pulley is used to overcome a resistance, as, for instance, it may be, to raise a weight.

This is done, as shown in the figure, by attaching a rope to the block of the pulley, and by passing another rope round the pulley, one end of this rope being fastened to a fixed point, and the acting force (or effort) applied to the other end.

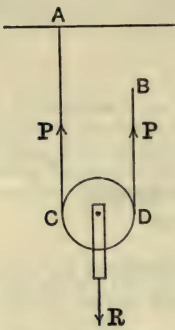

FIG. 119.—Single movable pulley.

Consider the simple case in which the two parts of the rope passing round the pulley are parallel to one another.

Let **P** be the applied force, and **R** the resistance overcome.

Since the pulley is in equilibrium under the action of the force **R** acting one way, and two forces each equal to **P** acting the other way, we have

$$2\mathbf{P}=\mathbf{R}.$$

$$\mathbf{P}=\frac{\mathbf{R}}{2}.$$

The mechanical advantage of the single movable pulley is 2.

Displacements of Pulley and of Rope.—If the pulley is displaced in the direction in which **P** acts through a distance d, that is, so as to overcome the force **R** through a distance d, the point of application of the force **P** must be displaced through a distance $2d$. For, we may suppose first that the pulley is displaced without rotating, and so that the same portion CD of the rope remains in contact with it, the part AC becoming slack. The end B then is displaced by d. Now let the pulley rotate, and let AB become tight again. A length d of rope goes over from AC to DB, and the end B is displaced through another distance d, that is, by $2d$ in all.

This result is of great use in applying the principle of work to the action of pulleys.

In the case we have just considered, if the pulley is displaced through a distance d, the work done against the resistance is $\mathbf{R}d$.

The work done by the force applied is $\mathbf{P} \, . \, 2d$.

These are equal.

$$\therefore \; 2\mathbf{P}=\mathbf{R}.$$

Systems of Pulleys.—Movable pulleys may be combined in a variety of ways to raise a weight or overcome a resistance.

The accompanying three figures indicate the three ways usually described in which this may be done.

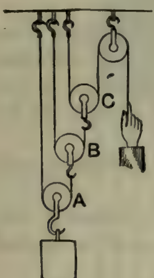

FIG. 120.

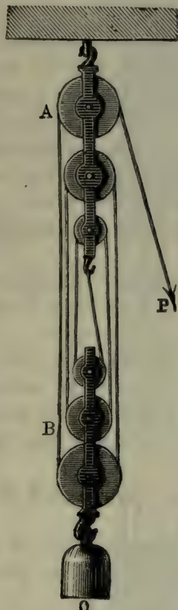

FIG. 121.

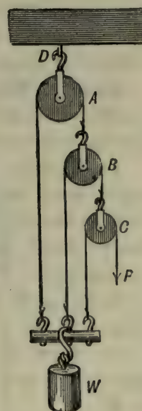

FIG. 122.

First, second, and third systems of pulleys.

These three methods of arranging pulleys are called the *first, second, and third systems.*

They are best illustrated by means of the figures, but they may also be described in words as follows.

First System.—In the first system a separate string passes round each pulley, one end of the string being fixed and its other end being attached to the next pulley, until we come to the last string, one end of this being fixed and the force being applied at the other end. The weight is attached to the block of the first or lowest pulley.

Second System.—In the second system the pulleys are fixed in two blocks, one block being fixed and the other movable; the same string passes round all the pulleys; and the weight is attached to the lower block.

Third System.—In the third system one pulley is fixed; one end of each string is attached to the weight; the string passes round one pulley and its other end is attached to the next, until we come to the last string, one end of which is attached to the weight, and the force is applied to its other end.

It should be noticed in the figures that the third system is just like the first turned upside down.

In any case the strings are all supposed to be parallel unless otherwise stated.

How to Work Examples.—In any examples on the use of pulleys it is advisable to work from first principles, and not to attempt to remember any formulae.

To illustrate the methods in the cases which have just been mentioned, with the figures drawn, we will determine the relations between **P**, the applied force, and **W**, the weight of the body lifted, supposing in every case that the pulleys are all weightless.

Suppose **W** to be the weight and **P** the force applied in each case.

(1) Since the tension throughout a string is constant, the tension in each part of the string to which the effort is applied is **P**.

$\therefore$ upward pull on pulley C is $2\mathbf{P}$.

$\therefore$ tension in each part of string passing round pulley B is $2\mathbf{P}$.

Upward pull on pulley B is $2 \cdot 2\mathbf{P} = 4\mathbf{P}$.

$\therefore$ tension in each part of string passing round pulley A is $4\mathbf{P}$.

Upward pull on pulley A is $2 \cdot 4\mathbf{P} = 8\mathbf{P}$.

$$\therefore \mathbf{W} = 8\mathbf{P}.$$

(2) The same string goes round all the pulleys, and the tension in each part of it is **P**.

Hence, since there are six segments of string going upwards from the lower pulley-block, the upward pull on this block is $6\mathbf{P}$.

$$\therefore \mathbf{W} = 6\mathbf{P}.$$

This is the most important combination of pulleys, and the one most frequently used in practice. Fig. 123 shows a slightly

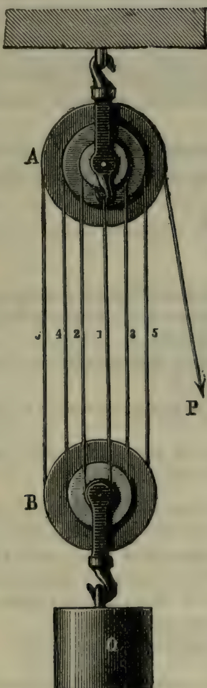

Fig. 123.

Fig. 124.

Pulleys arranged according to second system.

different arrangement of the pulleys, and Fig. 124 shows the arrangement generally employed in practice. The principle is the same in both arrangements.

(3) Tension in string to which the effort is applied is **P**.

$\therefore$ pulley *C* is pulled down by force 2**P**.

$\therefore$ tension in string passing round pulley *B* is 2**P**.

So tension in string passing round pulley *A* is 4**P**.

$\therefore$ the weight is sustained by the forces **P**, 2**P**, 4**P**.

$$\therefore \mathbf{W} = 7\mathbf{P}.$$

Same Results obtained by Principle of Work.—These results may all be easily obtained by the principle of work. For example consider the first case.

Suppose the weight to be drawn up through a distance *d*.

The pulleys *B* and *C* must rise through distances 2*d* and 4*d*, respectively.

The force **P** must act through a distance 8*d*.

Thus the work done by the force is 8*d***P**; and that done on the weight is *d***W**

$$\therefore 8d\mathbf{P} = d\mathbf{W}.$$

$$\mathbf{W} = 8\mathbf{P}.$$

Experiment 30.—Take a smooth pulley and attach a weight to it. Pass a string under the pulley. Attach one end of the string to a fixed point and the other to a spring-balance.

Lift the balance, and support the pulley and weight so that the two segments of the string are both vertical.

Find the mean force indicated by the balance necessary to sustain the weight.

This should be about half the weight of the pulley and body it supports.

Experiment 31.—Combine two or more movable pulleys according to the first system, taking care to have all the strings parallel.

Raise the end of the string at which the effort is applied till the weight is raised through one inch, or other suitable distance; and notice also the distance through which the effort end is raised.

From these observations obtain the velocity ratio.

Compare this with the velocity ratio deduced theoretically.

Experiment 32.—Take a pair of pulley-blocks arranged as in the second system. Attach a weight to the lower block. The size of this will depend on the size of the block, the strength of the rope, and the way the upper block is supported. If the blocks are of good size, such as are used for practical purposes, the weight may be as much as 200 lbs. or more.

Pull the rope at the effort end till the weight begins to rise and all the ropes are tight. Now continue to further pull down the rope till the weight rises through a considerable distance, which may be measured pretty accurately, say 2 or 3 feet, if the arrangement allows this to be done. Measure this distance, and measure the distance by which the rope has been pulled down at the effort end.

Thus determine the velocity ratio.

This should be a whole number and equal to the number of segments of rope connecting the two blocks.

EXPERIMENT 33.—With the same arrangement use a spring balance to determine the effort required to raise the weight. Thus find the mechanical advantage.

Find the efficiency from the fact that

efficiency = mechanical advantage ÷ velocity ratio.

Questions in which the weights of the pulleys are to be taken into account may be solved either by considering the equilibrium of the pulleys or by the principle of work. As an illustration of each method consider the first of the following questions.

EXAMPLE.—Three pulleys are arranged according to the first system to sustain a weight of 1 cwt.; the pulley to which the weight is attached weighs 6 lbs., and the other two, in order, 5 and 2 lbs. Find the force.

Let D, E, F be the fixed ends of the strings.

The rope DB balances a force of 118 lbs.' weight.

∴ tension in each part of this rope is 59 lbs.' weight.

Rope EC balances 59 + 5, *i.e.*, 64 lbs.' weight.

∴ tension in each part of this rope is 32 lbs.' weight.

Rope FP balances 32 + 2, *i.e.*, 34 lbs.' weight.

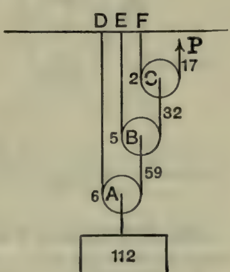

FIG. 125.

∴ the force **P** is **17 lbs.' weight.**

Alternative Method.—Suppose the weight to be raised through a distance d.

The work done on it is $112d$.

The pulleys A, B, C are raised through d, $2d$, $4d$.

$$\therefore \text{ the work done on them is } d\,.\,6+2d\,.\,5+4d\,.\,2.$$

The force, **P** lbs.' weight, acts through $8d$.

$$\therefore \text{ the work done by it} = 8d\,.\,\mathbf{P}.$$

$$\therefore\ 8d\mathbf{P} = d\,.\,112 + d\,.\,6 + 2d\,.\,5 + 4d\,.\,2,$$
$$8\mathbf{P} = 112 + 24,$$
$$\mathbf{P} = 17.$$

Example.—A pair of pulley-blocks has two pulleys in each block. Each block with its pulleys weighs 20 lbs. A body weighing 200 lbs. is sustained by the lower block. The rope after passing round all the pulleys is carried down and attached to the body. What is the tension in the rope?

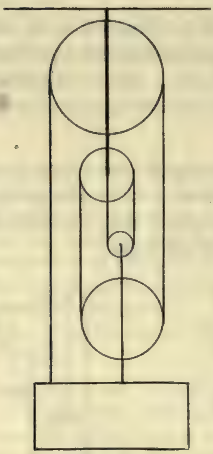
Fig. 126.

In this case the weight is sustained partly by the lower block and partly by the rope which comes from the upper block and is attached to it, as shown in the figure. So that there are 5 ropes sustaining the weight and the lower block.

Thus if **T** lbs.' weight is the tension throughout the rope,

$$5\mathbf{T} = 200 + 20,$$
$$\mathbf{T} = 44.$$

The required tension is **44 lbs.' weight.**

The Differential Pulley.—This apparatus is very frequently employed to lift heavy weights.

Two pulleys of different diameters are rigidly fastened together and turn on the same axis at A.

At B is a single movable pulley attached to the weight.

An endless chain passes over the two wheels at A and under the pulley B, and has a part hanging loose, as shown in the figure.

The wheels at A are not provided with smooth grooves; but the grooves are made to fit the chain so that the chain cannot slip in them. This may be done by having projections on the rim to work in the links as with ordinary bicycle chain-wheels. But more often the links fit into depressions cut in the groove to receive them.

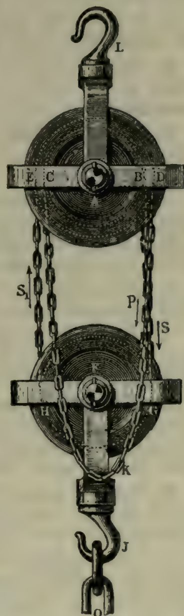

FIG. 127.—Differential pulley.

Suppose that a and b are the radii of the larger and smaller wheels at A. Let an effort **P**, acting as shown in the figure, be required to sustain a weight **W**.

We shall suppose that the two parts of the chain passing under B are practically parallel. Then the tension in each of these is $\frac{W}{2}$.

The double pulley A is in equilibrium under the action of the three forces **P**, $\frac{\mathbf{W}}{2}, \frac{\mathbf{W}}{2}$.

Thus, by moments about its axis, we have

$$\mathbf{P} \cdot a + \frac{\mathbf{W}}{2} \cdot b = \frac{\mathbf{W}}{2} \cdot a.$$

$$\therefore \quad \mathbf{P} \cdot 2a = \mathbf{W}(a-b);$$

$$\frac{\mathbf{W}}{\mathbf{P}} = \frac{2a}{a-b}.$$

EXPERIMENT 34.—Carry a heavy weight by means of a differential pulley. Draw down the chain by a considerable amount measuring the distance through which the effort acts. This may be done by measuring with a 2-foot rule as the chain comes down, or by having the whole endless chain marked off in known lengths all the way round and counting these as they pass down. Measure the distance by which the weight rises. Thus calculate the velocity ratio.

With a spring balance, or by means of weights, attached to the chain, measure the effort. Thus, find the mechanical advantage and the velocity ratio.

Exercises X. a.

1. In the first system, if there are four pulleys what force is required to raise a hundredweight?

2. In the first system, if there are four pulleys what weight can be raised by a force of 20 lbs.' weight?

3. With 6 movable pulleys arranged according to the first system, what is the ratio of the force applied to the weight of the body sustained?

4. With the first system of pulleys a force of 2 lbs.' weight is observed to balance a weight of 64 lbs. How many movable pulleys are there?

5. What is the smallest number of movable pulleys that must be arranged in the first system to enable a force of 10 lbs.' weight to raise a hundredweight? And what is the greatest weight that the given force could then sustain?

6. With the second system, if there are 3 pulleys in each block, find the force required to raise 2 cwts.

7. With the second system, if there are 3 pulleys in the upper and 2 in the lower block draw the arrangement and find the force required to raise 100 lbs.

8. With the second system there are 2 pulleys in the upper and one in the lower block, the lower block weighing 2 lbs. What force is required to raise a body weighing 50 lbs.? And what downward pull has the upper block to sustain?

9. If there are seven segments of string between the pulley blocks, what weight can a force of 4 grams' weight sustain. Show the arrangement in a diagram.

10. Two pulley blocks each having two pulleys are arranged so that the applied force acts vertically upwards. Draw a figure of the arrangement. Neglecting the weights of the blocks and pulleys, show that the force required to sustain a body is $\frac{1}{5}$th of the weight of the body, and the pull on the upper block $\frac{4}{5}$ths of the weight.

11. Four movable pulleys are arranged so that a separate rope passes round each one, and each rope has one end attached to a fixed point. All the ropes are vertical and the pulleys weightless. What force in the last rope will support a weight of 100 lbs. attached to the lowest pulley?

12. Two pulleys, each weighing 10 lbs., are arranged according to the first system to support a weight of 400 lbs. What force is required?

13. A body is raised by means of a system of pulleys arranged (a) according to the first system, (b) according to the third system, no fixed pulley being used in either case. Show that the pull on the fixed beam or support is greater in the second than in the first

case by the sum of the forces required to be applied in the two cases.

14. A body of weight W_1 is attached to a pulley of weight W_2. The pulley is carried by a rope of which one end is fixed, and the other end, to which the force is applied, is at a height l above the pulley. The rope weighs w units per unit of length. Show that the work done in raising the body by a height h is

$$h\{W_1 + W_2 + w(2l + h)\}.$$

15. What force is required to raise a block of stone weighing 280 lbs. with a pair of pulley blocks, each having three pulleys?

16. What is the greatest weight that can be lifted by a force of 40 lbs.' wt. with two pulley blocks, the upper one having 3 pulleys and the lower one having 2 pulleys, and weighing 10 lbs.?

17. Find the greatest weight that a man can lift, by standing on the floor and pulling downwards, with the help of a pair of pulley blocks, each having four pulleys, if the man weighs 150 lbs.

18. If the two parts of the rope supporting a weight by means of a perfectly frictionless pulley are not parallel, show that they must be equally inclined to the vertical.

19. One end of a string is attached to a fixed point and it passes under a smooth movable pulley and over a fixed one distant 10 feet horizontally from the fixed point. A weight of 10 lbs. is attached to the movable pulley; and this is allowed to hang 12 feet below the horizontal line through the fixed point and the fixed pulley. Find the tension in the string.

20. If by using a pair of pulley blocks, each weighing 16 lbs., a weight of 1 cwt. can be supported by a pull of 16 lbs.' wt., how many pulleys are there in each block?

21. In an arrangement of four pulleys, according to the first system, a pull of 14 lbs.' wt. sustains a weight of 176 lbs. The masses of the 1st, 3rd, and 4th pulleys, counting from the one attached to the weight, are 4, 3, 3 lbs. What is the mass of the 2nd?

22. Each of the four movable pulleys in the third system has the same weight; and a force of 4 lbs.' wt. is required to sustain a body weighing 202 lbs. What is the weight of each pulley?

23. In a differential pulley the radii of the two wheels are 6 and 8 inches. Show that the velocity-ratio is 8.

24. If in a differential pulley the radii of the wheels are 9 and 10 inches, and to raise a body weighing 200 lbs. a force of 15 lbs.' wt. must be used; what fraction of the work done is utilized in raising the weight?

25. If with any arrangement of pulleys a force of 10 lbs.' wt. acting through 17 feet raises 130 lbs. through 6 inches, what is the efficiency of the system?

26. With six pulleys arranged in two blocks, so that a single rope is used, it is found that a force of 14 lbs.' wt. is necessary to raise 160 lbs. What is the efficiency of the machine?

27. A pair of pulley blocks is working with an efficiency of ·4; and it is found that 120 grams' wt. is required to raise 4·8 kilos. How many sections of string are there between the blocks?

The Wheel and Axle is a machine consisting of a wheel firmly attached to a cylindrical axle, the two rotating about a common axis which is fixed in position.

To use the machine, a string passes round the wheel and another round the axle. A force applied to the string which passes round the wheel can in general overcome a much larger force acting at the string which passes round the axle.

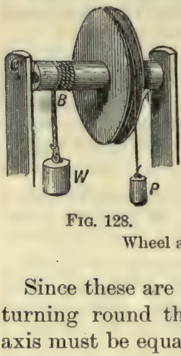

Fig. 128.

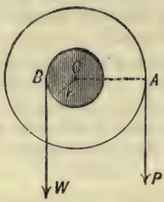

Fig. 129.

Wheel and Axle.

Fig. 128 gives a general view of the arrangement, and Fig. 129 is a diagram of it.

Let the radii of the wheel and axle be a and b.

Suppose a force **P** applied to the wheel sustains a weight **W** by means of the string passing round the axle.

Since these are in equilibrium, and the apparatus is capable of turning round the axis, the moments of **P** and **W** about the axis must be equal.

Thus, $$\mathbf{P} . a = \mathbf{W} . b,$$

or, $$\frac{\mathbf{W}}{\mathbf{P}} = \frac{a}{b}.$$

By the principle of work:

In a single turn the work done by **P** is $2\pi a\mathbf{P}$, and that done against **W** is $2\pi b\mathbf{W}$.

And these are equal. So that

$$a\mathbf{P} = b\mathbf{W}.$$

Sometimes a rope or string is passed round the wheel, touching it along a portion of its circumference, different forces being applied to the two parts of the string. In this case the friction between the rope and the wheel prevents the rope from slipping, although the forces in the two parts of it are unequal.

Let **T**, **T′** be the tensions in the two parts of the rope passing round the wheel, **T** being the greater, the other symbols having the same meanings as before.

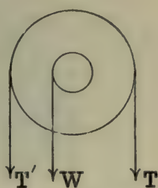

FIG 130. — Wheel and axle; wheel pulled by two segments of rope.

Then, by taking moments about the axis, we get

$$a(\mathbf{T}-\mathbf{T}')=b\mathbf{W}.$$

The same result would be got by considering that the work done by **T** in any motion, for instance, one complete turn, is equal to that done against **T′** and **W**.

Thus $$2\pi a\mathbf{T}=2\pi a\mathbf{T}'+2\pi b\mathbf{W}.$$

$$\therefore\ a(\mathbf{T}-\mathbf{T}')=b\mathbf{W}.$$

It should be noticed carefully that the friction which we have supposed to exist in this case, and which keeps the rope from slipping on the wheel, has no effect on the efficiency of the machine, because the force applied to the machine, and which we may regard as equal to $\mathbf{T}-\mathbf{T}'$, does no work against this friction. Work is only done against friction when one rough body is dragged over the surface of another against the resisting action of the friction force. Work would only be done against the friction in this case if the rope were to slip on the wheel, in which case the wheel would, as a rule, not rotate at all, and no *useful* work would be done.

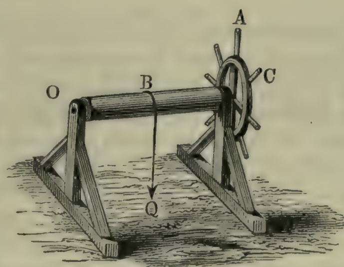

FIG. 131.—Windlass.

The Windlass. — The principle of the windlass is similar to that of the wheel and axle. A rope or chain *BQ* is wound round a drum *CO*, and this is turned by means of a handle, or sometimes a series of handles or spokes, as shown in the figure.

Thus, a weight at the end Q may be raised or other resistance may be overcome.

The relation between the forces may be easily obtained, as in the case of the wheel and axle.

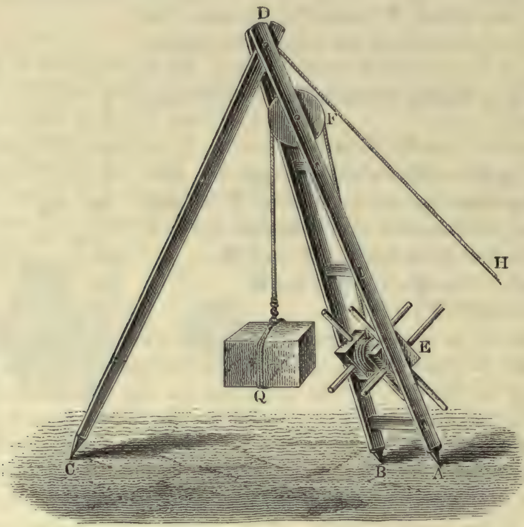

FIG. 132.—Derrick.

The Derrick.—Fig. 132 shows the arrangement known as the derrick, used for raising heavy bodies. It consists of a windlass and a fixed pulley supported by a tripod.

Differential or Chinese Windlass.—In this the drum consists of two cylindrical parts, B, C, of different radii. The weight is attached to a pulley A. The rope SS passes under this pulley from one side of the drum to the other. To raise the weight the drum is turned so that the rope is unwound from the smaller part of the drum and wound on the larger part.

Suppose the length of the arm, or crank, to which either of the handles D is attached, to be a, and the radii of the cylinders B and C to be b and c.

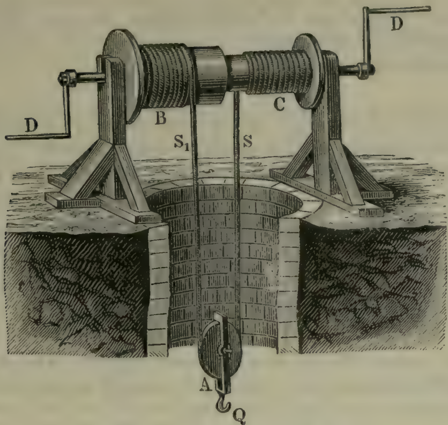

FIG. 133.—Differential windlass.

Suppose an effort **P** applied to the handle to sustain a weight **W**. Then the tension in the rope is $\frac{\mathbf{W}}{2}$. And by moments about the axis of the drum,

$$\mathbf{P}a = \frac{\mathbf{W}}{2}(b-c);$$

$$\frac{\mathbf{W}}{\mathbf{P}} = \frac{2a}{b-c}.$$

EXPERIMENT 35.—A wheel-and-axle is easily made by attaching a disc of wood to a cylinder of much smaller radius, or else by turning the whole in one solid piece. The first plan is simpler. A wheel of about 8 inches diameter and an axle of 1 inch diameter and 3 inches long will do very well. The wheel should be screwed firmly on the end of the axle with three or four screws, none of which passes through the centre. Two wire nails or screws may then be fixed at the centre of the wheel and at the other end of the axle to form an axis of rotation, and the arrangement set so that it can turn on two upright supports.

Fasten strings to the wheel and the axle, and wind them round in opposite ways.

First find the velocity ratio by finding how far the wheel string must be pulled down to raise a point of the axle string by a certain amount, say 2 or 3 inches.

Attach a weight to the axle string, and find what weight attached to the wheel string is just sufficient to raise the first. Thus determine the mechanical advantage.

Calculate the efficiency.

EXAMPLE.—A wheel is turned by means of a band passing round it, and kept from slipping by friction, the tensions in the two parts of the band being 10 and 40 kilos.' wt. The radius of the wheel is 2 metres. The machine is used to raise a load of 270 kilos. It is working at an efficiency of ·85. (*a*) What is the radius of the drum by which the load is raised? (*b*) In how many turns is the load raised through 20 metres?

(*a*) Let x metres be the required radius. The force that would be necessary to raise the load if the efficiency were 1 would be ·85 . 30 kilos.' wt.

$$\therefore \quad \cdot 85 \,.\, 30 \,.\, 2 = 270 \,.\, x.$$

$$x = \frac{1 \cdot 7}{9} = \cdot 1\dot{8}.$$

Radius of drum = **·1$\dot{8}$ metre.**

[The second part of the question may be easily solved by means of the value of the radius which we have just found. Or it may be solved independently as we shall show.]

Suppose n is the required number of turns.

The work done in n turns by the force 30 kilos.' weight is, since the circumference of the wheel is $\frac{44}{7} \,.\, 2$ metres,

$$n \,.\, 30 \,.\, \tfrac{44}{7} \,.\, 2 \text{ kilogram-metres.}$$

$\therefore$ useful work obtained is

$$\cdot 85 \,.\, n \,.\, 30 \,.\, \tfrac{44}{7} \,.\, 2 \text{ kilogram-metres.}$$

And work done in raising 270 kilos. through 20 metres is

$$270 \,.\, 20 \text{ kilogram-metres.}$$

$$\therefore \quad \cdot 85 \,.\, n \,.\, 30 \,.\, \tfrac{44}{7} \,.\, 2 = 270 \,.\, 20.$$

$$n = \frac{9 \,.\, 10 \,.\, 7}{\cdot 85 \,.\, 44} = \mathbf{16 \cdot 8 \text{ about.}}$$

Exercises X. b.

1. With the wheel and axle, if the diameter of the wheel is 2 feet and that of the axle 4 inches, what force is required to raise 150 lbs.?

2. A cylindrical drum is used to draw up a bucket of water weighing 60 lbs. It is turned by a handle at the end of an arm 15 inches long. If a force of 20 lbs.' wt. is to be applied to the handle, what must be the diameter of the drum?

3. What must be the radius of the wheel if a force of 7 lbs.' wt. is employed to draw up a hundredweight with an axle of $1\frac{1}{2}$ inch radius?

4. If the radius of the wheel is 3 feet, and the force applied is 12 lbs.' wt., how much work is done in two complete turns?

5. What work is done in each turn when the axle is 2 inches in diameter and the load raised is 40 lbs.?

6. The radii of a wheel and axle are 5′ and 6″. Some of the work is lost in friction. If a force of 125 lbs.' wt. is required to overcome a resistance of half a ton's wt., what is the efficiency of the machine?

7. A wheel and axle is working at an efficiency of $\frac{7}{8}$. The radius of the wheel is 1·2 metres, and a force of 7 kilograms' wt. is acting. How high is a load of 120 kilos. raised in 6 complete turns?

8. The efficiency of a wheel and axle is ·95. The radius of the wheel is 10 inches. If a force of 2 lbs.' wt. is used to raise a load of 10 lbs., what is the radius of the axle?

9. Show by a sketch a system of two pulleys, one fixed and one movable, one end of the cord being fastened to a fixed point in the beam which supports the machine.

If the angle between the parts of the cord, which supports the movable pulley, be 60°, find the power necessary to support a weight of 1,732 lbs. (Science and Art, 1899.)

The Screw.—The screw cannot be adequately described in a book. By observing any screw, such as one that is used by woodworkers, it will be seen that it consists of a cylinder, round which winds a spiral protuberance, or *thread*, the turns of which are uniformly spaced out from each other, and which intersects all straight lines that can be drawn on the surface of the cylinder, parallel to its axis, at a constant angle.

A screw used as a mechanical appliance is furnished with a bearing or block pierced with a hole having an inner thread into which the screw exactly fits. The block is fixed, and the threads on the screw and the block prevent the screw from moving bodily through the block without rotation. And, again, on the other hand, if the screw *is* rotated in the block the action of the threads is such as to compel it to move along in the direction of its length, as when a screw is turned in wood by means of a screw-driver, or a cork-screw is driven into or drawn out of a cork.

Screw-Jack.—The simplest case of a screw used as a machine is the screw-jack, which is employed for raising heavy weights. As shown in the figure, the screw is vertical and works in a block which remains fixed and is carried on a tripod stand. The screw presses upward against the body to be lifted, and is turned by means of a horizontal rod or handle: this causes the screw to rise and so to raise the body.

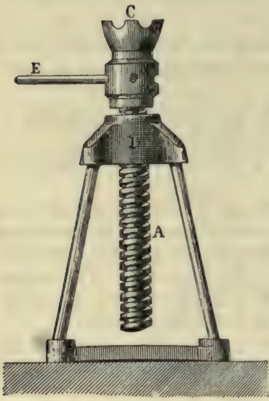

Fig. 134.—Screw-jack.

If the screw is turned so as to make it rise, and is rotated by just one complete turn, it rises through a distance equal to the distance between two consecutive turns of its thread, measured in a direction parallel to its axis. Each time that the handle, or arm, by which it is turned, comes back so as to be over and parallel to its original position, the screw has travelled along in the block by this distance.

The **pitch** of a screw is the distance between two consecutive turns of the thread measured parallel to the axis of the screw.

In the case of the screw, friction is very considerable. Of the force applied to the arm to rotate the screw and raise a weight, or overcome any other resistance, more than half, as a rule, is expended in overcoming the friction of the thread. That is, the force is more than twice as great as would be necessary if friction could be entirely eliminated. It should be noticed that here, as in all cases, the friction acts so as to oppose the motion. To move the screw against a resistance, the friction makes a much larger effort necessary; but if it is simply desired to hold the screw still against the action of a resistance the applied force may be made much smaller than would be necessary without friction. Indeed, in the case of the screw, the applied force could generally be entirely removed, the friction being sufficient to hold the screw at rest against the action of the force acting along the axis.

Work done by the force applied to the Arm.—Suppose a force **P** to be applied to the end of the arm, the arm being of length a. (a is taken to be the perpendicular distance from the point of application of the force to the axis of the screw.) And suppose that **P** acts in a direction at right angles to both the arm and the axis, or in other words **P** is at right angles to the arm and is in a plane containing the arm and at right angles to the axis.

Now, as the screw makes one complete turn, if it did not travel along its axis the point of application of **P** would describe a circle of radius a, and circumference $2\pi a$, and return to its old position. But, because the screw travels along its axis, the end of the arm really describes a helical curve, or the curve of a screw thread, which, for a single turn, is a little longer than $2\pi a$.

The projection of this curve on the plane at right angles to the axis in which **P** acts is the circle $2\pi a$; and it is this projection along which **P** acts throughout the motion; hence, the length of the circumference of this circle is the amount of displacement of **P**'s point of application *measured in the direction in which* **P** *acts.*

Therefore the work done by **P** in one turn is $2\pi a\mathbf{P}$.

Screw without Friction.—Imagine that we have an ideal screw in which there is no friction; and consider the relation that must exist in this case between the force applied and the resistance for equilibrium.

Let the pitch of the screw be d; the length of the arm a.

Suppose the screw is employed to raise a body of weight **W**.

Let a force **P** be applied to the arm in a direction at right angles to the arm and the axis of the screw.

The required relation is most readily found by the principle of work.

In a complete turn, the work done by **P** is $2\pi a\mathbf{P}$;

The work done against **W** is $d\mathbf{W}$.

$$\therefore \quad 2\pi aP = d\mathbf{W}.$$

$$\frac{\mathbf{W}}{\mathbf{P}} = \frac{2\pi a}{d}.$$

EXPERIMENT 36.—A useful form of screw for experiments is one which works vertically up and down in its fixed bearing, and carries, firmly fixed to its upper end, a wheel whose axis coincides with that of the screw. A weight to be raised may be placed on the wheel, and the wheel and screw may be rotated by a string passing round the edge of the wheel. With this apparatus the arm at which the effort acts is always the same, being equal to the radius of the wheel.

If this apparatus is not available a screw working vertically may be used with a horizontal arm. A weight to be raised can be tied on to the lower end of the screw.

Measure the pitch of the screw d, and the length of the arm at which the effort acts a; that is, the distance from the axis of the screw to the point at which the pull is applied.

The velocity-ratio is $\frac{2\pi a}{d}$.

Observe what pull is necessary to just cause the weight to rise.

Calculate the mechanical advantage and the efficiency.

If the screw is a strong one and firmly fixed, a considerable weight should be used.

Also find the efficiencies for various loads.

Toothed Wheels.—The Toothed Wheel which is frequently employed in mechanical appliances may be regarded as a modification of the wheel and axle. Suppose two wheels, each furnished with a series of uniform teeth round the rim, to rotate on parallel axes and to be so placed that neither can rotate without the other.

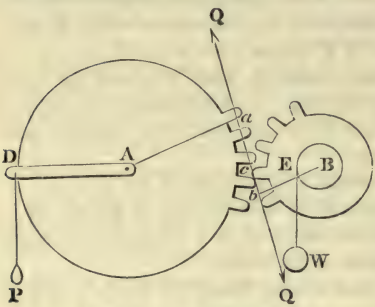

FIG. 135.—Toothed wheels.

Then, by the application of a force **P** tending to rotate one of the wheels, a resistance **W** tending to rotate the other can be overcome.

Let A, B be the centres of the two toothed wheels.

Let the forces **P** and **W** act at distances a and b from the axes of the wheels.

Let X be the point at which two teeth are in contact: And suppose that $AX = a'$, $BX = b'$.

Suppose that **Q** is the mutual action between the wheels at X; so that the wheel A presses the wheel B upwards at X with a force **Q**, and the wheel B presses the wheel A downwards at X with a force **Q**, **Q** being at right angles to the line AB.

Then by considering the equilibrium of the wheels we easily get:

$$\mathbf{P}a = \mathbf{Q}a',$$

$$\mathbf{W}b = \mathbf{Q}b'.$$

$$\therefore \quad \frac{\mathbf{W}b}{\mathbf{P}a} = \frac{b'}{a'};$$

$$\frac{\mathbf{W}}{\mathbf{P}} = \frac{ab'}{a'b}.$$

In the actual working of the wheels, when continuous motion takes place, the distances AX, BX will vary a little and the lines of application of the reaction **Q** will not always be at right angles to AB. But we may suppose that a', b' are the distances from the centres to the mean positions of the points of contact and that the mean position of **Q** is at right angles to AB.

Principle of Work applied to Toothed Wheels.—This is a question in which the principle of work may be advantageously employed. It will be noticed that in using this principle we do not have to take into account the intermediate force **Q**.

We shall suppose that as the wheels rotate and the point of contact between two teeth changes in position the ratio of **P** to **W** remains practically constant. This can be managed with well-cut wheels.

Suppose that there are m teeth on the wheel A and n teeth on the wheel B.

While A rotates once m teeth of B cross the line AB. Thus B performs $\frac{m}{n}$ of a complete turn.

P then acts through a distance $2\pi a$, and **W** is overcome through $\frac{m}{n} \cdot 2\pi b$.

Thus, equating the work done by **P** to that done against **W**, we get

$$\mathbf{P} \cdot 2\pi a = \mathbf{W} \cdot \frac{m}{n} \cdot 2\pi b.$$

$$\frac{\mathbf{W}}{\mathbf{P}} = \frac{n}{m} \cdot \frac{a}{b}.$$

These formulae for **W**/**P** are worked out as examples of the way in which questions of this sort should be solved.

m and n are proportional to the circumference of the wheels and therefore to their radii a', b'. Thus, we get the same result as before.

It would be possible to have m/n different from a'/b', which would seem to give a different result. But in such a case the wheels would not work well, and there would be no constant ratio between **P** and **W**. Suppose for instance the intervals between the teeth of A are considerably greater than those between the teeth of B. Then the rotation takes place with jerks, and as a tooth of A leaves one of B, before the next contact is made, A rotates by a small amount against no resistance, so that no force is then necessary to turn it.

The Winch.—This is a machine which is used for raising heavy weights or overcoming great resistances by means of a strong rope or chain.

The figure shows a winch which is worked by means of two handles F, F'. These turn a small toothed wheel, which works a much larger one HH. This is attached to a cylindrical barrel D, which turns with it and on which the rope is coiled.

Winches are also made with more than one pair of toothed wheels, that shown in the figure being a simple form.

It is clear that the mechanical advantage of the winch, supposing friction to be absent, will be found in just the same manner as has been done for a pair of toothed wheels.

The winch is an example of the way in which toothed wheels are employed to obtain a much greater force than that which is applied.

In clock- and watch-work trains, series of toothed wheels are employed to obtain a force much less than the force applied, but, at the same time, to obtain considerably increased motion ; for

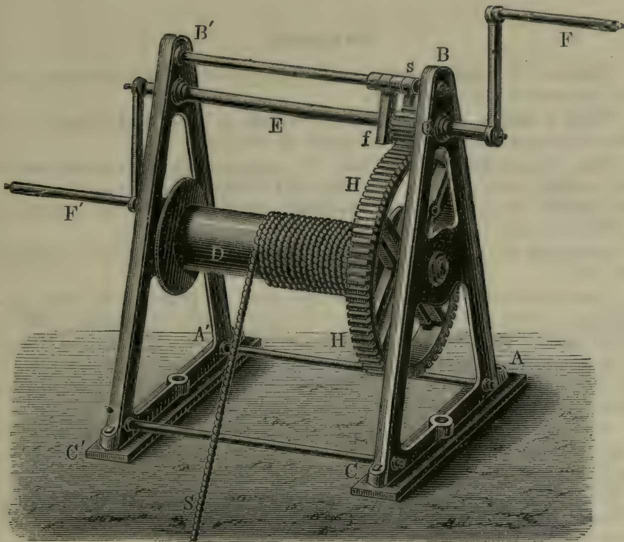

FIG. 136.—Winch.

the final wheel of the train rotates much more rapidly than the one to which the spring or weight is immediately applied.

EXAMPLE.—A winch is turned by a handle whose arm is 16 ins. long. This is attached to a wheel with 10 teeth, which works a wheel with 40 teeth. This wheel is attached to the barrel which is 4 ins. in diameter. The efficiency of the winch is 70 per cent. Find the force necessary to raise a weight of half a ton.

The force that would be required if no work were lost is

$$1120 \times \tfrac{10}{40} \times \tfrac{2}{16} \text{ lbs.' weight} = 35 \text{ lbs.' weight.}$$

But, since only 70 per cent. of the work done is utilized, the force required is

$$\tfrac{100}{70} \times 35 \text{ lbs.' weight} = \mathbf{50\ lbs.'\ weight.}$$

Summary.

Pulleys.—A **fixed pulley** is used merely to alter the direction and not the magnitude of a force.

With a **single movable pulley**, which is light and smooth, the effort is equal to half the resistance or weight.

Sets of movable pulleys may be grouped up in various ways. Any simple combination in which all the segments of string or rope are parallel may be treated by simple applications of the theory of parallel forces.

Questions on such combinations may also be solved by the principle of work.

The combination most commonly used in practice is that consisting of two pulley-blocks, each having several pulleys, with a continuous rope passing round all the pulleys.

In the **differential pulley** an endless chain passes over two wheels rigidly connected with each other and carried in a fixed block and under a movable pulley to which the weight is attached.

With the **Wheel and Axle** the relation between the forces may, in any case, be easily obtained either by taking moments about the axis about which the arrangement turns or by the principle of work.

In the case of the **Screw**, friction is of very great importance. The effort for a given resistance is very much greater, as a rule, than it would be if there were no friction.

On the supposition that there is no friction, the relation between effort and resistance is easily found by the principle of work, considering a single turn of the screw.

For two **Toothed Wheels** gearing into each other and fitting well the relation between effort and resistance, or between two forces tending to turn them about their axles, supposing that there is no friction, is obtained either by the principle of moments or by that of work.

Tooth wheels are applied in practice in winches and in clock- and watch-work trains.

CHAPTER XI.

THE BALANCE. STEELYARDS.

The Balance.—A *Balance* is an apparatus for determining the mass of a body.

The mass is determined by comparing it with the masses of bodies of definite known masses, for instance, pounds and ounces, or grams and fractions of a gram.

These standard bodies of definite masses are called *weights*; and the operation of comparing a body of unknown mass with them is called *weighing*.

The balance really determines in the first place which of the weights have the same *weight* as the given body.* Hence we infer that these weights have also the same *mass* as the given body; and thus we know the mass of the body.

Principle of the Balance.—In its very simplest form we may say that a balance is a straight uniform rod ACB, balancing about its middle point C, on a fixed fulcrum or support.

If two bodies of equal masses are hung on at A and B, the equilibrium of the balance will not be disturbed, the moments of the weights of these bodies about C being equal in magnitude.

And, if two bodies are hung on at A and B, and we find that the rod continues to balance about C; then, since the moments of the weights of these bodies about C must be equal, we infer that the weights, and therefore that the masses, of the bodies are equal.

* Note the two uses of the word *weight* in this sentence. It means

(*a*) A body of definite mass and weight used for weighing;

(*b*) The force of attraction of the earth on a body.

Description of a Balance.—A delicate balance, used for very accurate weighing, has the following arrangements. AB, which

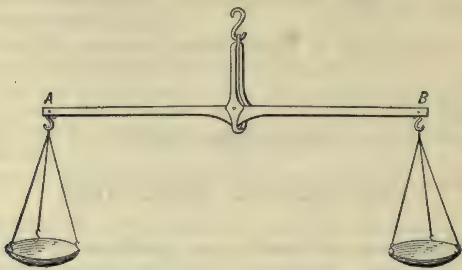

FIG. 136*a*.—A simple balance.

is called the beam, turns about a central support with as little friction as possible. This is accomplished by having an edge of hardened steel attached to the beam at right angles to it and turned downwards, and resting on a horizontal surface of very hard material, such as agate. This edge is called a *knife edge*.

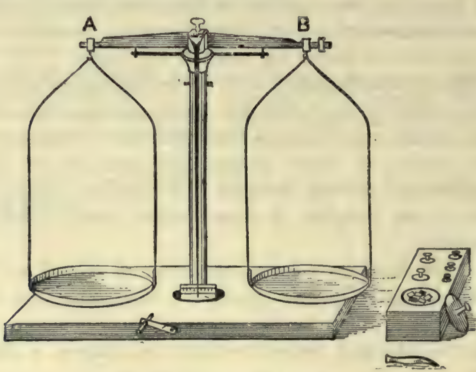

FIG. 136*b*.—A delicate balance.

Two receptacles for the body and the weights, called *scale-pans*, are hung on the beam at A and B, and are supported by two more knife-edges attached firmly to the beam at these points with their angles turned upwards. Hard plates rest horizontally on these edges, and from these plates the scale-pans hang.

The distances of the middle knife-edge from the other two are called the arms.

Double Weighing.—This is a method of using the balance in which we are quite independent of errors of adjustment in it; it being only necessary for accuracy that the balance should be sensitive, that is, that it should readily detect small differences in the mass of the body placed in either pan.

First Method. The body is placed in either pan and counterbalanced by other bodies, it may be standard weights or not. It is not necessary that the position of equilibrium to which the balance is brought should be its unloaded equilibrium position.

The body is thus removed and standard weights put in its place till the same equilibrium position is got again. The mass of the body is then the same as that of the weights which have replaced it.

Second Method. The body is placed in one pan and a body that will more than counterbalance it in the other. This body is called the *counterpoise.*

Equilibrium is restored by placing weights of mass M_1 in the pan with the body.

The body and M_1 are removed, and equilibrium is again restored by means of weights of mass M_2.

Then the mass of the body is $M_2 - M_1$.

This method is much more convenient than the other, because it is much easier to get the equilibrium state by means of the standard weights than with other bodies such as shot, sand, etc; and it is more convenient to do all the final adjustments for equilibrium by putting bodies in one pan only, say the right-hand one. The counterpoise may then be placed in the left-hand pan and left there throughout the operation; the body and the weights, M_1, M_2, being placed in the right-hand pan.

The method of double weighing is employed wherever great accuracy is required, as for instance in doing scientific work. But for all ordinary purposes a body is weighed by putting it in one pan and the weights in the other; and the balance is required to be so well adjusted as to allow this method to give results of fair accuracy.

In considering what conditions must be satisfied by a balance we shall suppose that the balance is to be used in this simple way.

Requisites of a good Balance.—(1) The balance should be *true*: that is, with any equal masses in the pans the balance should rest with the beam horizontal.

(2) The balance should be *sensitive* : that is, a small excess of mass on one side should produce an appreciable inclination of the beam.

(3) The balance should be *stable* : that is, it should quickly take up its position of equilibrium.

1. For the balance to be true : (i) It must rest, when not loaded, with the beam horizontal ; (ii) Any two equal weights in the pans must make the beam horizontal.

Hence, the moments of these weights about C must be equal; and therefore the arms must be equal. And it follows that any other pair of equal weights will make the beam horizontal.

When the balance satisfies the first of these conditions the second condition may be tested as follows. Put a body in one pan and counterbalance it, that is, put weights in the other pan so as to get the beam horizontal. Now interchange body and weights. If the beam is still horizontal the masses of body and weights must be equal, and the second required conditions for the balance obtains.

2. For the balance to be sensitive the middle knife-edge must not be much above the line of the other two, if at all, or far above the C.G. of the beam. The knife-edges should also work with very little friction.

3. For the balance to be stable, so that it may swing to its equilibrium position quickly, the middle knife-edge should be some distance above the line of the outer ones and above the C.G. of the beam.

Sensitiveness and Stability.—It is clear that sensitiveness and stability are opposed to each other, and, as a rule, the more sensitive a balance the less stable is it, and therefore the less quickly can we weigh with it. The best condition for the balance to be in will depend on the purpose for which it is to be used.

If it is to be used for scientific work in which great accuracy is required sensitiveness is essential, with a small amount of stability. Then the weighings can only be done slowly.

If it is a balance to be used for ordinary commercial purposes, stability, or quickness of indication, is necessary, with moderate sensitiveness. Then the weighings can be done quickly, but only with a small amount of accuracy as compared with what is attained in the other case.

For example, with a scientific balance it is by no means uncommon to weigh a body of 200 grams correctly to ·01 gram even in very rough work, which is an accuracy of 1 in 20,000; while for ordinary purposes it is generally sufficient to weigh a body of about 6 pounds correctly to half an ounce, which is about an accuracy of 1 in 200. Thus, the accuracy in the first case is 100 times as great as in the other.

Since the weights of the pans always act through the knife-edges A and B, in any problem about a balance we may consider the masses of the pans to be concentrated at A and B, and we may take the C.G. of the balance as that of the beam with masses equal to those of the pans concentrated at A and B.

Example.—With an inaccurate balance it is necessary to put 2 oz. in the left-hand pan, the other being empty, to make the beam horizontal, and 12 lbs. in the left-hand pan balances $12\frac{1}{4}$ lbs. in the other. What mass must be placed in the left-hand pan to balance 7 lbs. in the right?

Let a and b be the lengths of the left and right-hand arms.

Let W lbs. be the mass of the beam and pans; and h the distance of the C.G. of beam and pans (the masses of the pans being supposed collected on knife-edges) to right of fulcrum.

Let x lbs. be the required mass.

Then, by moments about the fulcrum,

$$\tfrac{1}{8}a = W \,.\, h, \quad\text{.........(1)}$$

$$12 \,.\, a = 12\tfrac{1}{4} \,.\, b + Wh, \quad\text{.........(2)}$$

$$xa = 7b + Wh. \quad\text{.........(3)}$$

From (1) and (2),

$$(12 - \tfrac{1}{8})a = 12\tfrac{1}{4}b.$$

From (1) and (3),

$$(x-\tfrac{1}{8})a=7b.$$

$$\therefore \quad \frac{x-\frac{1}{8}}{12-\frac{1}{8}}=\frac{7}{12\frac{1}{4}},$$

$$\frac{8x-1}{95}=\frac{28}{49}=\frac{4}{7}.$$

$$56x-7=380,$$

$$56x=387,$$

$$x=6\tfrac{51}{56}.$$

The required mass is **$6\frac{51}{56}$ lbs.**

In questions on balances, when it is said that a body in one of the pans *appears to weigh* a certain amount it is meant that weights of that amount must be put in the other pan to counterbalance it.

EXAMPLE.—A balance has unequal arms. A body of weight $\mathbf{W}$ when placed in one pan appears to have weight $\mathbf{P}$, and when placed in the other pan appears to have weight $\mathbf{Q}$. Show that $\mathbf{W}=\sqrt{\mathbf{PQ}}$.

Let a and b be the lengths of the arms.

Then, by moments about the fulcrum,

$$\mathbf{W}a=\mathbf{P}b,$$

$$\mathbf{Q}a=\mathbf{W}b.$$

$$\therefore \quad \frac{\mathbf{W}}{\mathbf{Q}}=\frac{\mathbf{P}}{\mathbf{W}}.$$

$$\therefore \quad \mathbf{W}=\sqrt{\mathbf{PQ}}.$$

EXAMPLE.—A balance has equal arms, but does not rest with its beam horizontal. A body, of weight $\mathbf{W}$, appears to have weight $\mathbf{P}$ or $\mathbf{Q}$ according as it is placed in one or the other pan. Show that $\mathbf{W}=\dfrac{\mathbf{P}+\mathbf{Q}}{2}$.

Let a be the length of each arm.

Let $\mathbf{w}$ be the weight of the balance with pans.

And let h be the horizontal distance from the fulcrum of the C.G. of the balance, including the pans, their masses being supposed concentrated on the points of support.

By moments about the fulcrum,

$$\mathbf{W}a = \mathbf{P}a + \mathbf{w}h,$$

$$\mathbf{Q}a = \mathbf{W}a + \mathbf{w}h.$$

$\therefore$, by subtraction,

$$(\mathbf{W} - \mathbf{Q})a = (\mathbf{P} - \mathbf{W})a,$$

$$2\mathbf{W} = \mathbf{P} + \mathbf{Q},$$

$$\mathbf{W} = \frac{\mathbf{P} + \mathbf{Q}}{2}.$$

Exercises XI. a.

1. A body of mass 4 when weighed in one pan of a balance with unequal arms appears to have mass $4\frac{1}{4}$. What is the ratio of the arms? and what will the body appear to weigh in the other pan?

2. If the arms of a balance are unequal, and a body weighs 14 and 15 oz. in the two pans, show that its real mass is 14·5 oz.

3. If the beam does not rest horizontal, and a body weighs 14 and 15 oz. in the two pans, show that its real mass is 14·491 oz.

4. Find the ratio of the arms when a body appears to weigh 400 and 441 grams in the two pans.

5. The arms of a balance are in the ratio 9 : 10. Two pounds of a substance are weighed, one in each pan; what is the apparent sum of the masses?

6. The arms of a balance are in the ratio 7 : 8. If a tradesman sells actually equal quantities of a material from the two pans, show that he gains $\frac{100}{112}$ per cent. If he sells apparently equal quantities from the two pans (that is, by weighing them out against equal weights in the pans), show that he loses $\frac{100}{113}$ per cent.

7. Tea is sold from the longer arm of a balance, the ratio of the arms being 15 : 16, nominally at 2s. 6d. per pound. What price per pound is actually paid?

8. The combined apparent weights of two pounds weighed in the two pans of a balance with unequal arms are $2\frac{1}{72}$ lbs.' weight. What is the ratio of the arms?

9. A balance has unequal arms, and it does not rest when unloaded with its beam horizontal. It is necessary to put a weight, w, into one pan to make the beam horizontal. A body, of weight W, appears to have weight P or Q according as it is placed in this or the other pan. Show that

$$W^2 - wW = PQ - wP.$$

10. If the arms are a and b, a being greater than b; and if w must be put in the pan on the arm a, show that the only weight that will be weighed correctly in either pan is equal to $\frac{aw}{a-b}$.

The Common or Roman Steelyard.—Suppose AOB is a rigid bar resting horizontally on a fulcrum at O, which is near the end A. The C.G. of the bar is between O and B. **P** is a movable weight which can be hung on the bar between O and B.

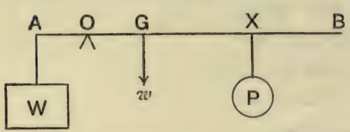

FIG. 137.—Principle of common steelyard.

Suppose a body of weight **W** is suspended from A. This can be balanced by placing **P** in a suitable position for any magnitude of **W** between certain limits. **W** will have its smallest value when **P** is placed as near to O as possible, and its greatest value when **P** is placed as near to the end B as possible.

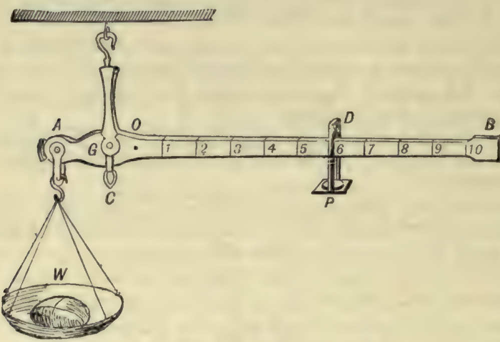

FIG. 138.—Common steelyard.

The position of **P** when a given weight **W** is balanced may be determined and may be made to indicate the magnitude of **W**.

This is the principle of the steelyard. This apparatus consists of a rod or bar AOB turning on a knife-edge support at O. A hook or pan carried on another knife-edge attached to the rod at A, and having its edge turned upwards, serves to support the body to be weighed. The movable weight **P** is attached to a ring which slides along the arm OB. This arm is graduated to show the values of **W** corresponding to the various positions of **P**.

Fig. 138 shows the actual apparatus in one of its forms.

Graduation of the Steelyard.—Suppose the steelyard, with the weight of the hook or pan supposed concentrated at the edge at A, to have mass **w** units and C.G. at G. (Fig. 137.)

Suppose a body of mass **W** to be balanced by placing the movable weight, of mass **P**, at X.

We shall find the relation that must exist between **W** and OX, so that we may know what mass is balanced with **P** in a given position, or, conversely, where **P** must be put to balance a given mass.

By moments about O we have

$$\mathbf{W} \cdot OA = \mathbf{w} \cdot OG + \mathbf{P} \cdot OX.$$

$$\therefore \quad \mathbf{W} = \frac{\mathbf{w} \cdot OG}{OA} + \frac{\mathbf{P}}{OA} \cdot OX.$$

By means of this relation we can determine the values of OX corresponding to various values of **W**, supposing **w**, OG, and OA to be known; and then marks may be made along the bar at the various positions of X, and the corresponding values of **W** marked against them.

As a simple example, suppose the mass of the steelyard is 10 lbs., that of P is 8 lbs., OG is 2 inches, and OA is 4 inches, and it is required to graduate the bar in pounds.

The above relation becomes, calling OX x inches,

$$\mathbf{W} = \frac{10 \cdot 2}{4} + \frac{8}{4} \cdot x;$$

$$\mathbf{W} = 5 + 2x.$$

If $\qquad \mathbf{W} = 10, \quad x = 2\tfrac{1}{2};$

if $\qquad \mathbf{W} = 11, \quad x = 3;$

if $\qquad \mathbf{W} = 12, \quad x = 3\tfrac{1}{2};$

and so on.

The bar being graduated in pounds, the graduations are, in this case, spaced out at half-inch distances.

It may not be possible to get **P** nearer to O than $2\tfrac{1}{2}$ inches Then the graduations begin with 10 lbs.

Suppose, in this case, that the greatest distance at which **P** can be placed from O is 20 inches. The corresponding value of **W** is given by

$$\mathbf{W} = 5 + 2 \cdot 20;$$

$$\mathbf{W} = 45.$$

The greatest weight that can be weighed is 45 lbs.

It is clear from the relation between the variable quantities W and OX and the fixed quantities, **w**, OG, OA, that in any case equal increments in **W** correspond to equal increments in OX. That is, the graduations are always uniformly spaced out.

This being known it is a much easier matter to graduate the steelyard in practice than by actually measuring the quantities **w**, **P**, OG, OA and using the formula.

Moreover the practical method is more likely to give accurate results than when the values found for these quantities are relied upon for the graduation.

The method is this. Find by experiment the positions of **P** for two known values of **W**, that is, by using for **W** two different standard weights. Divide the bar between these two positions into as many equal divisions as there are units of difference in the two values of **W**. And continue the divisions on both sides of these two determined positions.

Thus suppose we find that to balance weights of 5 and 25 kilograms **P** must be placed, respectively, at points S and T on the bar, and the distance ST is 85 centimetres. Divide ST into 20 equal parts, 85/20 or $4\frac{1}{4}$ cms. long each. The marks from S to T correspond to 5, 6, 7, . . . , 23, 24, 25 kilograms. Continue the divisions, each equal to $4\frac{1}{4}$ cms, on both sides of S and T and put the corresponding marks 4, 3, . . . , and 26, 27, . . . against them.

In graduating in this manner, two standard weights that are considerably different from each other, about the lowest and highest that the steelyard will weigh, should be used for the sake of accuracy; as any small error in determining the position of **P** for either of them will then make a smaller proportional error in the length of a division.

Experiment 37.—Take an ordinary steelyard and weigh the movable weight. Even if it cannot be removed this can be managed by so holding the bar of the steelyard above the pan in which the weight is put, that when the weighing is made it does not touch the weight or the ring which carries it. Call this weight **P** lbs.

Measure the length of a division corresponding to one pound. This is best done by measuring the length of the whole of the divided part, and dividing by the number of divisions. Let this length be d inches.

Measure as accurately as possible the distance from the fulcrum to the edge on which the load, or body to be weighed, is carried. Let this be a inches.

Compare the values of a and $\mathbf{P}d$, which should be equal to each other.

EXPERIMENT 38.—With the steelyard use a different movable weight. This may be done by attaching to the proper weight an additional one.

Take three known masses which can all be balanced by the new weight, and such that one is midway in value between the other two, such as 3, 5, and 7 pounds.

Balance these in turn by the new weight.

Show that the distance between the first and second positions is equal to that between the second and third.

The Danish Steelyard.—This is a bar AB having a weight or mass of metal attached to one end B, so that the C.G., G, is near to B. The fulcrum is movable. The body to be weighed is hung on the end A.

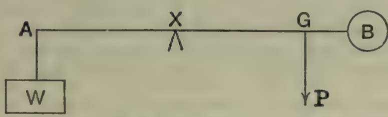

FIG. 139.—Principle of Danish steelyard.

The weight of the body is determined in this case by the position of the fulcrum under the rod, or the point of the rod which must rest on the fulcrum, and each graduation is marked with a number showing the weight of the body when that mark comes over the fulcrum.

It is clear that the zero graduation mark coincides with G, because when no weight is attached the rod balances at G.

Let X be the position of the graduation mark for a weight $\mathbf{W}$, so that the bar balances at X. By moments about G,

$$\mathbf{W} \cdot AX = \mathbf{P} \cdot GX;$$

i.e.,

$$\mathbf{W}(GA - GX) = \mathbf{P} \cdot GX;$$

$$GX = \frac{\mathbf{W} \cdot GA}{\mathbf{W} + \mathbf{P}}.$$

It follows from this that the graduations are not equally spaced out along the bar.

Exercises XI. b.

1. A steelyard weighs 16 lbs., and its C.G. is 3 inches from the fulcrum. The weight is hung on at 4 inches from the fulcrum; and the movable weight is 8 lbs. Find the graduation marks for 56 lbs. and 112 lbs.

2. A steelyard weighs 6 kilos., its C.G. is $8\frac{1}{2}$ cms. from the fulcrum; the point of support of the weight is 6 cms. on the other side of the fulcrum, and the movable weight is 3 kilos. Find the distance from the fulcrum of the 50 kilo. graduation, and the distance between successive kilo. graduations.

3. If P units is the mass of the movable weight, and the body to be weighed is hung on at a distance a from the fulcrum, show that the common distance between successive graduations for units of mass is $\frac{a}{P}$.

4. The body to be weighed is hung on at 4 inches from the fulcrum, and the graduations for 6 and 11 lbs. are 10 inches apart. What is the movable weight?

5. The steelyard and movable weight weigh 9 lbs. together. The steelyard will balance in the ring of the movable weight when this is at the 6 lb. graduation, and when the movable weight is at the 30 lb. graduation the two together will balance on the 14 lb. graduation. What are the masses of the steelyard and of the movable weight?

6. The weight of a Danish steelyard is 10 lbs., and the body to be weighed is hung on at 24 inches from the Centre of Gravity. How far from the Centre of Gravity are the graduations for 1, 2, and 3 lbs.?

7. On adding a certain amount to the body weighed, the movable weight must be moved by a distance a. If the movable weight is used with an extra pound weight added to it, on adding the same quantity as before to the body weighed, the movable weight must be moved by a distance b. Show that the mass of the movable weight is $\frac{b}{a-b}$ lbs.

8. If O is the fulcrum and G the C.G. of the bar, W and P the weights of the steelyard and movable weight, show that the graduation for any weight is at a distance proportional to that weight from a point C in the bar, where $OC = \frac{w \,.\, OG}{P}$, and C is on the opposite side of O from G.

9. A balance rests with its beam horizontal when unloaded, and its arms are 10 and $10\frac{1}{4}$ inches long respectively. What mass placed in the pan at the end of the longer arm will balance 15 lbs. placed in the other pan? (Camb. Jr. Loc., Mech., 1898.)

Summary.

A **Balance** is an apparatus for determining the mass of a body, by comparing the weight of the body with the weights of standard masses, used for weighing, and called **weights**.

Double Weighing is a method used for determining masses with great accuracy. By its means we are made independent of the adjustments, but not of the quality or sensitiveness, of the balance.

In a balance used for ordinary purposes, all the adjustments are supposed to be sufficiently exact to render double weighing unnecessary.

Requisites of a good Balance.—

(1) It should be *true* ;

(2) It should be *sensitive* ;

(3) It should be *stable.*

Sensitiveness and stability are opposed to each other. The best condition for the balance with regard to these qualities will depend on the purpose for which it is to be used.

The **Common or Roman Steelyard** is an apparatus for weighing bodies by means of a movable weight which can be placed on a graduated arm. The theory shows that equal distances along the arm correspond to equal increments in the weight of the body weighed. The steelyard is best graduated by using two standard weights, in turn, in place of the body to be weighed, and finding the corresponding positions of the movable weight and then dividing up the bar with graduations to correspond to these two positions.

The **Danish Steelyard** is a bar with a fixed weight at one end. The body to be weighed is hung on at the other end. The weight is determined by the point of the bar, that must rest on a fixed fulcrum or support that it may balance, graduations corresponding to various weights being marked along the bar.

CHAPTER XII.

VELOCITY. ACCELERATION. KINEMATICAL EQUATIONS.

The **velocity** of a moving body or point means the distance it passes over per unit of time.

We shall consider the motion of a body along a straight line, that is, *rectilinear motion.*

The velocity of a body moving along a straight line may be either *uniform* or *variable.*

Uniform velocity.—Suppose a body to move in such a manner that the distance passed over in any interval of time is always proportional to the length of the interval; then the velocity is uniform.

We may also say that the distances passed over in equal intervals of time must be equal, however short the intervals may be. It is necessary to specify this last condition; because a body may, for instance, move fast at the beginning of each second and slowly at the end of it, but so that the same distance is traversed in each second. The velocity would not be uniform; and shorter equal intervals than seconds could be taken, so that the distances in the intervals would not be equal.

Variable Velocity.—The velocity of a body is variable if it does not pass over a distance in any interval of time proportional to the time; or if equal distances are not described in all equal intervals of time.

Measure of Velocity.—To measure a velocity we require a unit in terms of which to measure length, and a unit in terms of which to measure time. The velocity is measured by the number of units of length passed over per unit of time. This definition is applicable, as we shall see, to any velocity, whether uniform or variable.

A velocity may be measured for instance in feet per second. The velocity of the body would then be measured by the number of feet passed over per second.

If the velocity is uniform the same number of feet is passed over in each second of the motion; and the velocity is measured by the number of feet passed over in any one second; or, in general, by the number of units of length passed over in any unit of time.

If the velocity is variable it must be measured in a different manner. We can then only speak of the velocity at any given instant of time because it is changing from instant to instant.

The measure of a variable velocity at a given instant of time is an example of what is called a *limiting value*, and the manner in which it may be found is illustrated in the following example.

Estimation of Variable Velocity.—Suppose a body moves in such a manner that the number of feet passed over in any given number of seconds is four times the square of the number of seconds. Let us consider how the velocity at the end of the 2nd second of the motion would be measured.

Distance in 2 seconds $=4 \,.\, 2^2=16$ feet.
Distance in 3 seconds $=4 \,.\, 3^2=36$ feet.
$\therefore$ distance in 3rd second $=20$ feet,
and mean velocity in 3rd second $=20$ feet per second.

In same way we should find that mean velocity in 2nd second $=12$ feet per second.

These two results are considerably different from each other, and are both of them very different from what we want to find, which is some quantity between the two.

To find a closer approximation to the actual velocity at the end of the 2nd second we shall now take a much shorter interval than 1 second, namely ·1 second.

Distance in 2 seconds $=4 \,.\, 2^2 = 16$ feet.
Distance in 2·1 seconds $=4 \,.\, (2{\cdot}1)^2=17{\cdot}64$ feet.

∴ distance in interval ·1 second after end of 2nd second is 1·64 feet,

and mean velocity in this interval is 16·4 feet per second.

If we take an interval ·01 second after end of 2nd second we find in the same way that mean velocity during it is

$$\frac{4(2\cdot01)^2 - 4 \cdot 2^2}{\cdot01} = 16\cdot04 \text{ feet per second.}$$

An interval ·001 second would give for mean velocity 16·004 feet per second.

As we take shorter and shorter intervals we get nearer and nearer to the quantity 16 feet per second. And by taking a short enough interval we can make the mean velocity as near to 16 feet per second as we please.

This then is the limiting value which we are seeking. And the velocity at the end of the 2nd second is 16 feet per second.

The same thing may be quickly shown by the help of Algebra.

Let t second be a very short interval of time, t being a small fraction.

$$\begin{aligned} \text{Distance in 2 seconds } &= 4 \cdot 2^2 \text{ feet.} \\ \text{Distance in } (2+t) \text{ seconds} &= 4(2+t)^2 \text{ feet} \\ &= 4(4+4t+t^2) \text{ feet.} \end{aligned}$$

∴ distance in the interval t second after end of 2nd second is

$$4(4t+t^2) = 16t + 4t^2.$$

∴ mean velocity in the interval t second is

$$\frac{16t+4t^2}{t} = 16 + 4t \text{ feet per second.}$$

By taking t smaller and smaller this approaches 16 as its limit.

∴ the velocity at the end of 2 seconds is 16 feet per second.

The value of the velocity at any instant of the motion is specified as follows.

Take a very short interval of time, t, including the instant in question (t will be a very small fraction, and the given instant may be at the beginning or end of this interval). Let s be the distance passed over in this time. Then s/t is the mean velocity during the interval t; or it is the velocity which the body must have throughout this interval in order to describe the distance, s, that is actually described.

The velocity of the body *at* the instant in question is the limiting value of the fraction s/t, when t, and consequently also s, is made indefinitely small.

In other words, it is the limiting value of the mean velocity in an indefinitely short interval, including the instant.

The measure of the velocity at a given instant, when the velocity is varying, may also be specified as the number of units of length which the body would pass over in a unit of time if it went on moving for a whole unit of time with the velocity which it has at that instant.

Acceleration.—When the velocity of a body, or the rate at which it moves, becomes greater, we say, in ordinary language, that the velocity is accelerated, or is undergoing acceleration; and if the velocity becomes less we say that it is undergoing retardation. But, in mechanics, the word *acceleration* is applied to any change of velocity; it only differs in algebraical sign according as the velocity is actually becoming greater or less.

The acceleration of a moving body at any instant means the rate at which its velocity is increasing.

Or it is *the increase of velocity per unit of time.*

If the velocity is actually increasing, it follows that the acceleration is a positive quantity; if the velocity is decreasing the acceleration is negative.

If the velocity of a body at any instant is given in feet per second, its acceleration would be specified by saying that a velocity of so many feet per second is added to its velocity per second. Or, in short, we should say that it has an acceleration of **so many feet per second per second.**

Thus, if at a given instant a body's velocity is 4 feet per second, and at the end of a second it is 6 feet per second, and at the end of another 8 feet per second, and so on, the velocity increasing uniformly; we should say that its velocity increases by 2 feet per second in every second, or that its acceleration is 2 feet per second per second.

Uniform Acceleration is the acceleration of a body whose velocity increases by equal increments in equal intervals of time, however small the intervals of time may be taken.

Or, it is the acceleration of a body in which the velocity increases in any time by an amount proportional to the time, whatever be the time considered.

When this is not the case the acceleration is variable.

In the cases we shall have to consider, the acceleration will generally be uniform.

An operation that is very frequently necessary in dealing with the motion of bodies is to express a velocity or an acceleration, that is given in one system of units, in terms of another system.

This operation is called **conversion of velocities or of accelerations.** We shall give some examples of it.

EXAMPLE.—A train is moving at the rate of 60 miles per hour. Express this velocity in feet per second.

Velocity of 60 miles in 1 hour

$=$ velocity of 60×5280 feet in 60×60 secs.

$=$ velocity of $\dfrac{60 \times 5280}{60 \times 60}$ in 1 sec.

$=$ velocity of 88 feet per sec.

It is rather useful to remember that a velocity given in miles per hour is converted to feet per second by multiplying by the fraction $\frac{22}{15}$; and, conversely, to convert from feet per second to miles per hour we must multiply by $\frac{15}{22}$.

To remember which is which, note that 22 occurs in the numerator for converting from miles to feet, because it occurs as a factor in the number of feet in a mile.

EXAMPLE.—Express an acceleration of 32 feet per second per second in yards per minute per minute.

Velocity of 32 ft. per sec. is added in each sec.

$\therefore$ vel. of $\frac{32}{3}$ yds. per sec. is added in each sec.

$\therefore$ vel. of $\frac{32}{3} \times 60$ yds. per min. is added in each sec.

$\therefore$ vel. of $\frac{32}{3} \times 60 \times 60$ yds. per min. is added in each min.

That is, required acceleration is

$\frac{32}{3} \times 60 \times 60 =$ **38400 yds. per min. per min.**

The solution may be more shortly written as follows :

Acceleration of 32 ft. per sec. per sec.

=accn. of $\frac{32}{3}$ yds. per $\frac{1}{60}$ min. per $\frac{1}{60}$ min.

=accn. of $\frac{32}{3} \times 60$ yds. per min. per $\frac{1}{60}$ min.

=accn. of $\frac{32}{3} \times 60 \times 60$ yds. per min. per min.

=accn. of 38400 yds. per min. per min.

We have seen that, when feet and seconds are the units of length and time, a velocity is specified in feet per second and an acceleration in feet per second per second. Again, we may speak of a velocity or of an acceleration as so many units of velocity or acceleration *in feet and seconds*, or as so many *foot-second units* of velocity or acceleration.

Thus, a velocity of two feet per second is 2 units of velocity in feet and seconds, or 2 foot-second units of velocity.

An acceleration of 2 feet per second per second is 2 units of acceleration in feet and seconds, or 2 foot-second units of acceleration.

Example.—A body starts from rest, and has an acceleration of 7 cms. per sec. per sec. When will it be moving with a velocity of 1 metre per minute ?

Vel. of 1 metre per min.

=vel of 100 cms. in 60 secs.

=vel. of $\frac{5}{3}$ cms. per sec.

Vel. acquired in each second is 7 cms. per sec.

∴ time required to acquire vel. of $\frac{5}{3}$ cms. per sec.

$=\frac{5}{3} \div 7$ secs. $=\frac{5}{21}$ **sec.**

Example.—A body is moving with a velocity of 8 feet per second ; and it comes to rest in $1\frac{1}{2}$ minutes. Express its acceleration in centimetres and seconds. 1 ft.=30·48 cms.

[Note that it is best to first express all quantities in terms of the units required in the answer.]

Given velocity of body $= 8 \times 30{\cdot}48$ cms. per sec.

$= 243{\cdot}84$ cms. per sec.

This vel. is lost in 90 secs.

$\therefore$ vel. is lost at rate of $\frac{243{\cdot}84}{90}$ cms. per sec. in each sec.

$\therefore$ the acceleration is negative, and is equal to

$-2{\cdot}709\dot{3}$ cms. per sec. per sec.

Exercises XII.

1. Show that a velocity given in miles per hour is converted to yards per minute by multiplying by the fraction $\frac{88}{3}$.

2. A cyclist rides at the rate of 12 miles per hour. In what time does he travel 100 yards?

3. A body travels $16t^2$ cms. in t seconds. What is its mean velocity in the 4th second, and its velocity at the end of 4 seconds?

4. A train starts from rest and gets up a velocity of 40 miles per hour in 3 minutes. Express its acceleration in feet and seconds.

Motion with uniform Acceleration.—The case of a body moving in a straight line with uniform acceleration is one of great importance and must be considered at length.

Symbols used.—We shall, in general, in all formulae and questions, use the following symbols in the senses specified.

u denotes the initial velocity of a body, that is, not necessarily the velocity with which it begins to move, but the velocity which it has at the beginning of any observed or contemplated portion of its motion.

t denotes the time for which it is moving for any contemplated portion of its motion.

a denotes its acceleration.

s denotes the distance through which it travels.

v denotes the velocity it possesses at the end of the contemplated portion of its motion.

Convention with regard to Signs.—The quantities u, v, a, s, which are quantities having direction as well as magnitude, we shall always suppose to be measured in one definite direction, in one sense along the line of motion (generally in the sense in which the body is moving). If one of these quantities happens, in any case, to be in the other sense, the corresponding symbol will be of the negative sign.

Thus, if a body is moving along a straight line, and we agree to measure the quantities in the direction and sense of its motion; and if its velocity is becoming 3 feet per second less in each second, we should say that its acceleration is -3 feet per second per second.

The equations connecting these symbols, that is, the equations referring to the pure motion of a body, without any reference to the cause of it, are called **kinematical equations** or **formulae.**

Relation between u, v, a, t.

Since the velocity is increased by a units in each unit of time,

$\therefore$ in t units of time it is increased by at units.

But it is u units to begin with.

$\therefore$ in t units of time it becomes $u+at$ units.

i.e., $$v=u+at. \qquad (1)$$

Relation between s, u, v, t.

Since the velocity increases uniformly its *time average* (or its mean value) is half-way between its initial and final values, that is, it is $\frac{u+v}{2}$.

The distance described is the same as that which would be described in the same time with the mean velocity. Hence we have

$$s=\frac{u+v}{2}\cdot t.$$

These statements, although they are quite correct, hardly constitute an exact proof; because it has not been proved that the distance which is described is the same as that which would be described with a velocity equal to the time average of the velocity.

Accordingly the following exact, but more troublesome, proof is given.

For clearness, suppose v greater than u.

The proof will be quite similar if v is less than u.

Imagine the time t divided with a large number, n, of equal intervals, each equal to $\frac{t}{n}$.

Let the velocities at the beginnings of these intervals be $v_1, v_2, v_3, \dots v_n$.

v_1 is the same as u.

The $n+1$ quantities u, v_2, v_3, ... v_n, v increase successively by equal increments, so that v_2 is as much above u as v_n is below v.

$$\therefore\ v_2+v_n=u+v.$$

So
$$v_3+v_{n-1}=u+v.$$

And so on.

Now the entire distance is greater than the distance that would be described if the body moved throughout each interval with the velocity it has at the beginning of the interval; and it is less than it would be if the body moved throughout each interval with the velocity it has at the end of the interval.

That is, s is greater than

$$(u+v_2+\ \dots\ +v_{n-1}+v_n)\frac{t}{n};$$

and it is less than

$$(v+v_n+\ \dots\ +v_3+v_2)\frac{t}{n}.$$

$\therefore$ the error made by taking either of these for s is less than their difference $(v-u)\frac{t}{n}$.

By taking n large enough we can make this quantity as small as we please. Hence the true value of the distance is got from either of the above expressions by making n indefinitely great; or it may be got from their mean value by making n indefinitely great.

Now the mean value of these two expressions, got by combining the first terms in each, then the second terms, and so on, is

$$=\left(\frac{u+v}{2}+\frac{u+v}{2}+\ \dots\ n\ \text{times}\right)\frac{t}{n},$$

$$\because\ v_2+v_n=u+v,\ v_3+v_{n-1}=u+v,\ \text{and so on.}$$

$$\therefore\ \text{mean value}=\frac{u+v}{2}\,.\,t.$$

This must be the correct value for the distance when n is made indefinitely large; and it remains the same whatever n may be.

Hence the actual value of the distance is given by

$$s=\frac{u+v}{2}\,.\,t.$$

Relation between u, s, a, t.

This is easily obtained from the last, or may be obtained independently in the same way as the last.

Since $$v=u+at,$$

and $$s=\tfrac{1}{2}(u+v)t,$$

$$s=\tfrac{1}{2}(2u+at)t;$$

$$s=ut+\tfrac{1}{2}at^2. \quad \ldots\ldots(2)$$

Relation between v, s, a, t.

This is obtained in much the same way as the last.

Since $$v=u+at,$$

or $$u=v-at,$$

and $$s=\tfrac{1}{2}(u+v)t,$$

$$s=\tfrac{1}{2}(2v-at)t$$

$$=vt-\tfrac{1}{2}at^2. \ldots$$

Relation between u, v, s, a.

$$v=u+at.$$

$$\therefore\ v^2=u^2+2uat+a^2t^2$$

$$=u^2+2a(ut+\tfrac{1}{2}at^2),$$

$$v^2=u^2+2as. \quad \ldots\ldots(3)$$

The three equations marked (1), (2), (3) are of very great importance. They are now collected for convenience of reference.

$$\mathbf{v}=\mathbf{u}+\mathbf{at}, \quad \ldots\ldots(1)$$

$$\mathbf{s}=\mathbf{ut}+\tfrac{1}{2}\mathbf{at}^2, \quad \ldots\ldots(2)$$

$$\mathbf{v}^2=\mathbf{u}^2+2\mathbf{as}. \quad \ldots\ldots(3)$$

The two others which have not been numbered are sometimes useful, but they are of less importance. They may be called (4) and (5).

$$\mathbf{s}=\tfrac{1}{2}(\mathbf{u}+\mathbf{v})\mathbf{t}, \quad \ldots\ldots(4)$$

$$\mathbf{s}=\mathbf{vt}-\tfrac{1}{2}\mathbf{at}^2. \quad \ldots\ldots(5)$$

These we will call **the 5 kinematical equations.**

The equations (1), (2), (3) should be most carefully learned. They are all that are necessary for doing questions about rectilinear motion with uniform acceleration. Some questions may be more quickly done with the help of (4) and (5). But any such question can be done by means of (1), (2), and (3) alone. It is, however, convenient to know (4) and (5), and they are easily remembered, (4) because of its simplicity, as it merely states that distance is equal to the product of mean velocity and time, and (5) by means of its similarity to (2).

It should be noticed that several questions on formula No. (1) have already been done, these questions being very easily done by simple arithmetic, it being merely necessary to remember clearly what acceleration means. The connexion between the entire velocity, the original velocity, the time, and the acceleration, or rate of gaining velocity, is now expressed in an algebraical formula.

It has been seen that (2), (5), and (3) are not algebraically independent ones, but are derived from (1) and (4).

Other equations are sometimes given referring to cases in which the acceleration is in the direction opposite to that in which s, u, and v are measured. But it is recommended to use the equations given above only, and to let the sense of the acceleration be in all cases indicated by sign.

Consider, for example, the following question.

EXAMPLE.—A body begins to move with velocity 40 feet per second, and has an acceleration of 3 feet per second per second opposite to the direction of motion; how far does it go in 4 seconds?

Here we should use the equation (3), and take s feet to be the required distance. The acceleration is a negative quantity, so that $a = -3$.

$$\begin{aligned} s &= ut + \tfrac{1}{2}at^2 \\ &= 40 \cdot 4 + \tfrac{1}{2}(-3) \cdot 4^2 \\ &= 136. \end{aligned}$$

Required distance is **136 feet.**

Again, if the body starts from rest, $u = 0$, and special formulae may be obtained from those given above by writing $u = 0$ in them.

Those corresponding to (1), (2), (3) are

$$v = at,$$

$$s = \tfrac{1}{2}at^2,$$

$$v^2 = 2as.$$

These should, however, not be regarded as new formulae, but merely as particular cases of (1), (2), (3), readily obtained by putting the particular value 0 for u.

It should be noticed that in many books and questions the symbol f is used for the acceleration. A student should, therefore, be prepared to find f used in this way instead of a. It seems, however, more natural to use the symbol a for acceleration.

The other quantities, too, are not always denoted by the symbols which are used here. For example, V is often used for the initial velocity.

Summary.

The **velocity** of a body or point is the distance passed over per unit of time.

Uniform velocity is measured by the number of units of distance passed over in any unit of time.

Variable velocity at any instant of time is measured by the limiting value of the fraction got by dividing the distance passed over in a short interval of time, including the instant in question, by the interval, when the interval is made indefinitely small.

Acceleration means rate of increase of velocity, or increase of velocity per unit of time. In British units it is expressed as so many feet per second per second.

Uniform acceleration is that of a body whose velocity in any time increases by an amount proportional to the time.

For the motion of a body in a straight line with uniform acceleration the symbols u, v, t, a, s are used for initial and final velocities, time, acceleration, and distance.

One sense along the line of motion is taken as the positive sense, and each of the quantities u, v, a, s is reckoned positive if measured in this sense, and negative if measured in the opposite sense.

The equations connecting these quantities are called the **kinematical equations.** They are

$$\mathbf{v}=\mathbf{u}+\mathbf{at} \qquad \text{(1)}$$

$$\mathbf{s}=\mathbf{ut}+\tfrac{1}{2}\mathbf{at}^2 \qquad \text{(2)}$$

$$\mathbf{v}^2=\mathbf{u}^2+2\mathbf{as} \qquad \text{(3)}$$

$$\mathbf{s}=\tfrac{1}{2}(\mathbf{u}+\mathbf{v})\mathbf{t} \qquad \text{(4)}$$

$$\mathbf{s}=\mathbf{vt}-\tfrac{1}{2}\mathbf{at}^2 \qquad \text{(5)}$$

Of these (1), (2), (3) may be regarded as of primary importance, and (4) and (5) of secondary importance.

CHAPTER XIII.

USE OF KINEMATICAL EQUATIONS. ACCELERATION DUE TO GRAVITY.

General Hints.—Some general hints, which will probably be found useful, will now be given for the use of the equations.

In rough work jot down the symbols for all the quantities given and the quantity required, and consider what equation connects them. From this equation find the unknown symbol.

EXAMPLE.—The velocity of a body is 20 feet per second, and it has an acceleration in the direction of its motion of 8 feet per second per second. How far will it travel in 2 seconds?

[We know v, a, t. We want s. The required equation is No. 2.]

Let s feet be the required distance.

By $$s = ut + \tfrac{1}{2}at^2,$$
$$s = 20 \,.\, 2 + \tfrac{1}{2} \,.\, 8 \,.\, 2^2$$
$$= 56.$$

Distance is **56 feet.**

EXAMPLE.—A body is moving with a velocity of 10 cms. per second. It has an acceleration of 3 cms. per sec. per sec. opposite to the direction of its motion. In what distance will its velocity be 6 cms. per sec.?

[We know u, a, v. We want s.]

By $$v^2 = u^2 + 2as,$$
$$6^2 = 10^2 + 2(-3)s.$$
$$s = \frac{100 - 36}{6} = 10\tfrac{2}{3}.$$

Distance is **$10\tfrac{2}{3}$ cms.**

EXAMPLE.—A body starts from rest, and in 2 minutes acquires a velocity 8 kilometres per second. What is its acceleration?

2 minutes $=120$ secs.

8 kilometres per sec. $=8000$ cms. per sec.

[We know u, t, v. We want a.]

Let the acceleration be a cms. per sec. per sec.

By
$$v=u+at,$$
$$8000=0+a\,.\,120,$$
$$a=66\tfrac{2}{3}.$$

The acceleration is **$66\frac{2}{3}$ cms. per sec. per sec.**

This question could, of course, be easily solved without an algebraical formula.

After converting the time and the velocity we may go on, thus:

Velocity acquired in 120 secs. $=8000$ cms. per sec.

∴ vel. acquired in 1 sec. $=66\frac{2}{3}$ cms. per sec.

EXAMPLE.—After a body has been moving for 3 seconds it is observed to have a velocity of 9 feet per second and has travelled 18 feet. What is its acceleration?

[We know t, v, s. We want a. We use formula (5).]

Let a ft. per sec. per sec. be the acceleration.

By
$$s=vt-\tfrac{1}{2}at^2,$$
$$18=9\,.\,3-\tfrac{1}{2}a\,.\,3^2.$$
$$\frac{9a}{2}=9.$$
$$a=2.$$

The acceleration is **2 feet per sec. per sec.**

We shall now show how this may be done with (1), (2), and (3) alone. We shall first indicate how, in general, questions to which (4) and (5) apply may be solved by the use of (1), (2), and (3) only.

In rough working, jot down the symbols for all the quantities given and the quantity required. If no equation [*i.e.* among

(1), (2), (3)] is found to connect them, write down in a third place the symbol for another unknown quantity, such that two connecting equations can then be found.

Thus, taking the above example.

[We know t, v, s. We want a. Introduce u.

We may notice that, as there are only five symbols in the equations, and as t, v, s, a are not connected by any of the equations, u is certainly the symbol to be introduced.]

Let a ft. per sec. per sec. be the acceleration.

Let u ft. per sec. be the initial velocity.

By $$v=u+at,$$

and $$s=ut+\tfrac{1}{2}at^2,$$

$$9=u+3a,$$

and $$18=3u+\frac{9a}{2}.$$

$$\therefore\ u+3a=9,$$
$$2u+3a=12.$$

Solving, we get $$3a=6,$$
$$a=2.$$

What is done in a case of this sort is to write down two simultaneous equations in two unknown quantities, one of which is the quantity we require.

It was best not to use (3) because this contains u^2.

EXAMPLE.—A body has an acceleration of 4 cm.-sec. units opposite to the direction of its motion. At a certain instant it is moving with a velocity of 20 cms. per sec. When will it be at a point 42 cms. further on?

Let t secs. be the required time.

By $$s=ut+\tfrac{1}{2}at^2,$$

$$42=20t+\tfrac{1}{2}(-4)\,.\,t^2.$$
$$t^2-10t+21=0,$$
$$t=3, \quad \text{or} \quad =7.$$

The required time is **3 secs.** or **7 secs.**

The meanings of the two results obtained in this case should be carefully noticed. They are both correct, and both should be given.

Since the acceleration is opposite to the direction of motion the velocity diminishes, becomes zero, and then changes sign. The body now turns and passes through its old positions, and in 7 secs. from the observed instant it is at the same point as it was 4 secs. before, or at 3 secs. from the observed instant.

EXAMPLE.—A body has an acceleration of 7 foot-second units opposite to the direction of its motion. At a certain instant it is moving with a velocity of 12 feet per second. When and where will it stop?

Let t secs. be the time in which the body stops; and s feet the distance from the point at which it was observed.

Then, by $$v = u + at,$$
$$0 = 12 + (-7)t,$$
$$t = \tfrac{12}{7} = 1\tfrac{5}{7}.$$

By $$v^2 = u^2 + 2as,$$
$$0^2 = 12^2 + 2(-7)s,$$
$$s = \tfrac{144}{14} = \tfrac{72}{7} = 10\tfrac{2}{7}.$$

The body will stop in $1\frac{5}{7}$ secs. after the instant at which it is observed, and will have travelled $10\frac{2}{7}$ feet.

EXAMPLE.—A body starts with a velocity of 25 units and has an acceleration of 5 units. In the n^{th} unit of time after starting it is observed to move through $47\frac{1}{2}$ units of distance. Find n.

Distance in n units of time $= 25n + \frac{1}{2} \cdot 5 \cdot n^2$.

Distance in $(n-1)$ units of time $= 25(n-1) + \frac{1}{2} \cdot 5 \cdot (n-1)^2$.

$\therefore$ distance in n^{th} unit of time $= 25 + \frac{1}{2} \cdot 5(2n-1) = 5n + 22\frac{1}{2}$.

$$\therefore\ 5n + 22\tfrac{1}{2} = 47\tfrac{1}{2}.$$
$$n = 5.$$

Many questions occur which cannot be solved by a simple application of the equations to one part of the motion. The following is an example.

EXAMPLE.—A body starts from rest with uniform acceleration. After it has been moving for some time it is observed to travel 72 feet in 2 seconds; and it then has a velocity of 42 feet per second. How long was the body moving before the 72 feet began to be described?

Let t seconds be the time required.
Let a feet per second per second be the acceleration.

By
$$s = vt - \tfrac{1}{2}at^2,$$
$$72 = 42 \cdot 2 - \tfrac{1}{2} \cdot a \cdot 2^2.$$
$$2a = 84 - 72,$$
$$a = 6.$$

Velocity at the end of t seconds is $6t$ feet per sec.

$\therefore$ by
$$s = ut + \tfrac{1}{2}at^2,$$
$$72 = 6t \cdot 2 + \tfrac{1}{2} \cdot 6 \cdot 2^2,$$
$$12t = 72 - 12,$$
$$t = 5.$$

The body moves **for 5 seconds before it begins to describe the 72 feet.**

Notice that we have quoted the standard formula, $s = ut + \frac{1}{2}at^2$; but have employed t in a different sense in the working. This need introduce no confusion. The formula is only quoted that we may have before our eyes the necessary relation between the initial velocity in any part of the motion, the time of that part, etc.; and in the second part of the motion, to which we wish to apply this formula, we know that the time is 2 seconds, and that the initial velocity is $6t$ feet per second, t meaning, as stated, the number of seconds up to the beginning of this part.

EXAMPLE.—A body known to be moving with uniform acceleration passes a point A, and at 3, 5, and 8 seconds later it passes the points B, C, D. BC is 56 cms., and CD is 129 cms. With what velocity did the body pass A?

Let u cms. per sec. be the velocity at B, and a cms. per sec. per sec. the acceleration of the motion.

By the formula $s = ut + \frac{1}{2}at^2$, applied to the path BC,
$$56 = u \cdot 2 + \tfrac{1}{2}a \cdot 2^2;$$
or
$$u + a = 28, \quad \text{.................................(1)}$$

Since $BD = 185$ cms., and BD is passed over in 5 seconds, by the formula $s = ut + \frac{1}{2}at^2$,

$$185 = u \cdot 5 + \tfrac{1}{2}a \cdot 5^2;$$

or $$2u + 5a = 74. \quad\text{.......}(2)$$

Combining (1) and (2) we get

$$a = 6, \quad u = 22.$$

The body passes A 3 seconds before it passes B;

$$\therefore \text{ velocity at } A = (22 - 3 \cdot 6) \text{ ft. per sec.}$$

$$= \mathbf{4 \text{ feet per second.}}$$

Exercises XIII. a.

Find the uniform accelerations of the bodies in the following eight cases:

1. Velocity of 20 feet per second becomes 30 feet per second in 3 seconds.

2. Velocity of 20 cms. per second becomes 16 cms. per second in 8 seconds.

3. Starting from rest acquires velocity of 4 cms. per second in 8 minutes.

4. Has velocity of 10 yards per second and comes to rest in half an hour.

5. Velocity increased by 4 feet per second in every minute.

6. Velocity of 1000 feet per second lost in $\frac{1}{5}$ second.

7. Velocity of 60 miles per hour lost in 2 minutes.

8. Has velocity of 40 feet per second and comes to rest in $\frac{1}{14}$ second.

9. A body starts from rest with an acceleration of 6 feet per second per second. How long will it take to travel 48 feet?

10. A body is moving with a velocity of 6 cms. per sec. and an acceleration of 2 cms. per sec. per sec. opposite to the direction of its motion. Show that it will be 8 cms. further on in 2 or in 4 seconds. Explain the two results.

11. What must be the initial velocity of a body if it has an acceleration of 6 units and travels 81 units of distance in 3 units of time?

12. A diver strikes the water with a velocity of 24 feet per second, and ceases to sink at 4 feet below the surface. What is the mean retardation in the water of his downward velocity?

13. A carriage set rolling comes to rest in 36 feet, while it loses velocity at the rate of 1 foot per second in each second. What was its velocity at first?

14. A railway train slows down to rest with uniform retardation of its velocity in 80 seconds; and during that time it travels 2000 feet? At what rate was it going?

15. A boy starts to slide with a velocity of 12 feet per second and slides 24 feet. Find the acceleration of his motion, supposed uniform, and the time for which he is sliding.

16. A boat is observed to be moving in the water with a velocity of 15 feet per second. Its velocity falls off uniformly, and in 5 secs. it has travelled 50 feet. What is its velocity then?

17. A carriage on rails, slowing down uniformly, passes over 36 feet in 12 seconds. It then passes over 12 feet more before coming to rest. Show that its velocity at the beginning of the 36 feet was 4 feet per second.

18. A bullet traverses a 2 in. plank, and its velocity is changed in doing so from 1200 to 950 feet per sec. What is the mean rate of diminution of its velocity, and in what time does it pass through?

19. The velocity of a train is reduced from 60 to 30 miles per hour in 200 yards. In how many more feet will it come to rest?

20. A body moves through 7 feet in 2 seconds; and comes to rest in 3 seconds more. What is its acceleration?

21. A body moves through 50 cms. in 10 secs., and through 32 cms. in the next 10 secs. In how many more seconds will it come to rest?

22. A body starts from rest and moves through 90 cms. and in the next second it moves through $32\frac{1}{2}$ cms. How long has it been moving altogether?

23. A body moves through 11 feet in 3 seconds. 10 seconds elapse, and it moves through 50 feet in the next 3 seconds. If its acceleration has been uniform the whole time find it.

24. A bullet strikes a tree with a velocity of 1400 feet per second and lodges after penetrating 4 inches. What is its acceleration?

Acceleration due to Gravity.—It is found by experiment that any body moving under the action of gravity or the attraction of the earth alone, that is, falling freely under the action of its own weight, undergoes an acceleration which, in a given locality, is practically uniform. It varies a little from one part to another of the earth's surface, and also to a very slight extent with the altitude.

In practice, whenever a body is let fall, it is under the action of a resistance due to the air, and the effect that this produces on the acceleration due to gravity varies according to the size, shape, and density of the body. When the resistance of the air is got rid of, so that the acceleration is due to gravity alone, it is found to be, in one given place, precisely the same for all bodies.

Value of Acceleration due to Gravity.—The acceleration due to gravity, expressed in foot-second units of acceleration, varies from about 32·09 at the equator to 32·25 at the poles. In the latitude of London it is about 32·2. This expressed in centimetre-second units is about 981.

In formulae it is usual to express the value of the acceleration due to gravity by means of the symbol g, whatever may be the units used, or the particular value of the acceleration at the locality in question.

In numerical examples where no value is specified g is always to be taken as 32 when feet and seconds are used, and as 981 when centimetres and seconds are used.

In the kinematical equations, when they are used to refer to the motion of a body under the action of its own weight in a vertical straight line, if distances, etc., are measured downwards a must, of course, be replaced by g, or the number for which g stands; if distances, etc., are measured upwards, a must be replaced by $-g$.

EXAMPLE.—A body is let fall. How far does it descend in the first and in the second seconds of its motion?

By the equation $s = ut + \frac{1}{2}at^2$,

No. of feet in 1 sec. $= \frac{1}{2} . 32 . 1^2 = 16$;

No. of feet in 2 secs. $= \frac{1}{2} . 32 . 2^2 = 64$.

Thus distance in 1st sec. $=$ **16 feet;**

distance in 2nd sec. $=$ **48 feet.**

EXAMPLE.—A stone is thrown up with a velocity of 40 feet per second. In what time will it be 16 feet above the point from which it was thrown? Interpret the two results.

Let t seconds be the required time.

By
$$s = ut + \tfrac{1}{2}at^2,$$
$$16 = 40 . t - \tfrac{1}{2} . 32 . t^2$$
$$= 40t - 16t^2.$$

$$2t^2 - 5t + 2 = 0,$$
$$(t-2)(2t-1) = 0,$$
$$t = \tfrac{1}{2}, \text{ or } = 2.$$

The stone will be at the point in question **in ½ sec. and in 2 secs.**

At the end of half a second it will have first reached the point ; at the end of two seconds it will have ascended to its highest point, and will have just reached the given point on its way down.

EXAMPLE.—A body is thrown up with a velocity of 32 feet per second. How high will it rise?

The body will rise till its velocity is for an instant zero. After this it will fall.

Let s feet be the distance through which the body will rise.

By $$v^2 = u^2 + 2as,$$
$$0 = (32)^2 + 2(-32)s.$$
$$\therefore \quad s = \frac{(32)^2}{2 \cdot 32} = 16.$$

The body will rise **16 feet.**

The Case of a Body thrown upwards.—Suppose a body is thrown upwards with velocity V (any system of units being used).

Let s be the height to which it rises, and t the time taken in rising.

It rises till its velocity becomes 0.

$\therefore$ by $$v^2 = u^2 + 2as,$$
$$0 = V^2 + 2(-g) \cdot s.$$
$$s = \frac{V^2}{2g}.$$

By $$v = u + at,$$
$$0 = V + (-g)t,$$
$$t = \frac{V}{g}.$$

Let V' be the velocity on reaching the starting point, and t' the time taken in falling.

Then, since the body has to fall a distance $\frac{V^2}{2g}$, by

$$v^2 = 2as,$$

$$V'^2 = 2g \cdot \frac{V^2}{2g}.$$

$$V' = V.$$

And by

$$s = ut + \tfrac{1}{2}at^2,$$

$$\frac{V^2}{2g} = 0 + \tfrac{1}{2}gt'^2.$$

$$t'^2 = \frac{V^2}{g^2},$$

$$t' = \frac{V}{g} = t.$$

Thus, we have found expressions for the height to which the body rises, and the time of rising. But it is not recommended that an attempt should be made to commit these to memory. Practice in readily deducing them from the kinematical equations should be acquired.

It is, however, very useful to remember the two results, (i.) that the velocity on reaching the starting point is equal to that of starting, (ii.) that the time of coming down is equal to that of going up.

Exercises XIII. b.

1. A stone dropped from a cliff takes $4\frac{1}{2}$ secs. to reach the bottom. What is the height of the cliff?

2. A stone is thrown upward with a velocity of 30 metres per second. Show that in 4 seconds it will be exactly 41·52 metres above the point of projection. And show that the other time at which it is at the same point is about $2\frac{1}{4}$ seconds after the instant at which it was thrown up.

3. A body is thrown up with a velocity of 48 feet per second. How high does it rise?

4. A body is thrown up with a velocity of 48 feet per second. In what time does it reach the ground?

5. A body thrown downward descends 784 feet in 3 seconds. With what velocity was it thrown down?

6. A stone was thrown up and reached the ground in 4 seconds. How high did it rise?

7. A stone falls through 800 feet in one second of its motion. How long has it been falling before?

8. A stone is dropped into a well 56 feet deep. Sound travels at the rate of 1100 feet per second. In what time after letting the stone fall is the splash heard?

9. With what velocity must a body be thrown up so as to reach a height of 121 feet?

10. A bullet shot vertically upwards, and supposed to have uniform downward acceleration of 32 foot-second units, passes a point whose altitude is 5600 feet 5 seconds after it starts. For how long does it rise, and how high does it go?

11. A stone is dropped from the top of a tower 100 feet high, and at the same instant another is thrown up from the bottom with a velocity of 80 feet per second. When and where will they meet?

12. For how many seconds must a body have been moving from rest with uniform acceleration of 32 feet per second per second in order to describe 1840 feet in the next second?

13. How far has a body moved from rest with 8 cm.-sec. units of acceleration if it will pass over 132 cms. in the next second?

14. A stone is dropped from a height; and 2 seconds later another is thrown vertically down with a velocity of 128 feet per second. In what time will this overtake the first and at what depth?

15. A stone is dropped into a well and the splash is heard $3\frac{1}{2}$ seconds later. Sound travels at 1100 feet per second. Find, to the nearest foot, the depth of the well?

16. A body having a velocity of 4 units at a given point is under a constant acceleration of 2 units opposite to the direction of motion. How far will it be from this point when its velocity is 6 units in the direction of the acceleration?

17. A body moving with uniform acceleration passes over 6600 yards in 10 minutes; at the end of the distance it has a speed of 30 miles an hour. What was the speed at the beginning of the distance? (Coll. Precep., 1897.)

18. Find the height to which a body will rise when thrown vertically upwards with a velocity of 490 centimetres at a place where $g=980$. (Coll. Precep., 1898.)

19. A particle, moving from rest with uniform acceleration, passes over 38 feet in the 10th second of its motion. Over what space will it pass during the 15th second?

What is its average velocity during the 15th second? (Oxford Locals, 1899.)

20. A and B are two points in a vertical line; A being 64 feet above B. A body is let fall from A, and half a second afterwards another body is let fall from B; show that the former body will overtake the latter, and find at what distance below B it will do so. (Science and Art, 1897.)

21. A body falling freely has a velocity V at a certain instant; in three seconds from that instant it falls through a distance a, and in six seconds from that instant it falls through a distance b; if the ration of a to b equals that of 4 to 13, find V. (Science and Art, 1898.)

22. Write down the formula which expresses the time taken by a body, moving with uniform acceleration f, in passing over a space s from rest.

Neglecting the resistance of the air, find the time taken by a body in falling from the top of the Eiffel Tower, 300 metres high. (*N.B.*—A metre $=3{\cdot}281$ feet.) (Science and Art, 1898.)

23. Define acceleration and explain clearly what is meant by an acceleration of 5 feet per second per second.

Express an acceleration of 880,000 feet per hour per hour in miles per minute per minute. (Oxford Locals, 1897.)

24. Two bodies start from rest with uniform acceleration, and acquire a velocity of 400 yards per minute, one after travelling 400 yards, the other after moving for a minute. Find their accelerations in feet per second per second. (Oxford Locals, 1898.)

25. While a train is travelling at the rate of 40 miles an hour, the brakes are put on and the train is brought to rest with a uniform retardation after moving over a distance of $\frac{1}{8}$ of a mile. Find what time elapses before the train comes to rest, and what is the retardation. (Camb. Jr. Loc., Mech., 1896.)

Summary.

Questions involving the use of the kinematical equations may be solved by writing down the symbols for the quantities that are given and for the one required, and then selecting the equation which connects them.

Questions involving the use of equations (4) and (5) can also be solved by using (1), (2), and (3) only, introducing a symbol which is not given or required, and selecting two equations from (1), (2), and (3), connecting the required symbol and the introduced symbol twice over with the given quantities, and then solving.

The **Acceleration due to Gravity** is a uniform vertically downward acceleration.

In questions about the motion of bodies in vertical straight lines under the action of their weights, the kinematical equations should be used with the value of this acceleration instead of a; the + or − sign be used according as the quantities are measured downwards or upwards.

In the case of a body thrown vertically upwards, the following results should be remembered:

(i.) The velocity on reaching the starting point is equal to that of starting.

(ii.) The time of coming down is equal to the time of going up.

CHAPTER XIV.

DYNAMICAL MEASURE OF FORCE. NEWTON'S FIRST AND SECOND LAWS OF MOTION.

The Measure of a Force.—In Statics forces are compared with each other. The measure of a force is the number of times that it contains some standard force, such as the weight of a pound. And two forces are said to be equal if they balance each others' tendencies when acting in opposite senses along the same straight line.

In Dynamics a force is measured independently of other forces; it is measured by reference to quantities which are not forces; it is estimated by its effect in producing motion in a body.

In practice it almost always happens that when a body is moved in any way its weight causes some resistance to the motion. If the body is lifted the weight is directly opposed to the lifting effort; if it is pushed along on a horizontal surface the weight causing it to press on the surface produces a friction force that has to be overcome. But even if the weight could be entirely got rid of every body would offer a resistance to being set in motion.

Inertia.—It is a matter of common experience that it is more difficult to set a heavy body in motion than a light one. Imagine two carriages, a heavy one and a light one, on very good smooth wheels and placed on a very smooth level horizontal surface. It is harder to set the heavy one moving by a short quick push than the light one. The difference in the weights has something to do with this, it is true; because the difference in the weights causes a difference in the friction forces which oppose the motions. But the friction may be made extremely small and still a very obvious difference in the difficulty of setting the two in motion remains.

Again, if the two carriages are moving with the same velocity it is more difficult to *stop* the heavy one. This time the greater friction would *assist* the effort applied to the heavy carriage. But still a greater effort has to be applied to the heavier one.

A similar experiment may be tried by suspending two different masses at the ends of long strings. Suppose a 14-lb. weight and a 56-lb. weight be each suspended by a cord 20 feet long. If one of these be then pulled an inch or so away from its position of rest the horizontal pull necessary to hold it in its new position, that is, to overcome the statical action of its weight is very small indeed. And friction is almost completely eliminated. But it will be found much more difficult to move the 56 lbs. about horizontally by short sharp jerks than to do the same to the 14 lbs.

Thus, every body resists the action of a force tending to set it in motion, or to stop its motion, simply because it is a material body and without any reference to any other forces that may already be acting upon it and which have to be overcome.

The *inertia* of a body is the property in virtue of which it resists the action of forces tending to change its state of motion: or it is the measure of the difficulty of changing its state of motion.

The *mass* of a body is sometimes said to be the quantity of matter in it: it is the same as its inertia; or it is a measure of the difficulty of affecting the state of motion of the body. And the only direct way of measuring a mass by means of a standard of mass or of comparing two different masses with each other is by comparing the actions of forces on them.

We have seen that a body resists the action of force tending to set it in motion; and in the case of a body moving on a smooth surface we have seen that it resists the action of force tending to bring it to rest. If left alone its velocity would gradually fall off and it would come to rest. If the surface were made smoother, so that the force opposing the motion is diminished, it would move for a longer time, and would ultimately come to rest. The smaller we make the opposing force the more slowly does the velocity change. In practice it is impossible to entirely remove the opposing force; in an experiment, even if the surface could be made perfectly smooth, so as entirely to get rid of friction, we should have the resistance of the air acting on the body; but by getting rid of all resistances as far as possible we can make the rate at which the velocity of the body falls off very small indeed.

This leads us to expect that, if a body could be set in motion and entirely removed from the action of all forces, it would continue to move in just the same manner for ever. This is expressed in the following:

Newton's First Law of Motion.—Every body continues in a state of rest or of uniform motion in a straight line, except in so far as it is compelled to change that state by external impressed force.

We have seen that the direct evidence for this is not complete and cannot be made so. It is impossible to experiment on a body completely removed from the action of forces; as far as direct evidence goes we can only say that the more nearly we approach the conditions contemplated in the law the nearer do we get to a state of motion with uniform velocity.

Evidence on which Newton's Laws rest.—But we may say for this and for the other two laws of motion that the chief evidence on which they rest is indirect. Especially from Astronomical observations is this evidence obtained. The laws of the motions of the heavenly bodies are based on Newton's laws of motion, and could not hold unless these were true. And continual observations on these bodies are always proving with what exactness the laws are obeyed.

Relation between Force acting and Mass moved.—We shall now consider more particularly the relation between the force acting on a mass, the mass and the way in which it is moved.

If a force acts on a mass entirely removed from the action of other forces, it gives to it a velocity which increases as long as the force acts, and is proportional to the time for which the force acts. That is, the force produces in the mass a constant *acceleration.*

If the force is doubled the acceleration will be doubled; if trebled, trebled; and so on. That is, for a given mass, the force is proportional to the acceleration which it produces.

If another mass is found in which the same force as acted on the first produces the same acceleration, this mass must be considered equal to the first. And if the two masses be put together so as to obtain one double of the first, the force necessary to produce a given acceleration must be doubled. If the mass is trebled the force must be trebled, and so on. That is, the force is proportional to the mass, when a given acceleration is always produced.

Putting these two results together we see that the *force* is proportional to the product *mass* × *acceleration*.

This result may be put in another way by using a new term, of which the definition will now be given.

Momentum.—The momentum of a body is the product of its mass and its velocity.

Thus if the mass is m units and the velocity v units, the units in terms of which the quantities are measured being any whatever, the corresponding measure of the momentum is mv.

Now suppose at the same time that the body has an acceleration a. Its increase of the momentum in time t is

$$m(v+at)-mv=mat.$$

∴ rate of increase of momentum, or increase per unit of time, is ma.

We can thus write the result that force is proportional to the product of mass and acceleration in the following form.

Force is proportional to the rate of change of momentum which it produces.

This is expressed in the following :

Newton's Second Law of Motion.—Rate of change of momentum is proportional to the acting force, and takes place in the direction in which the force acts.

We have not yet fixed upon any method of measuring mass, or any unit in terms of which to measure it. For this we may use the pound. Suppose at the same time that we use the foot and second as units of length and time. Let m and a be the mass and acceleration of a body measured in these units. Then the force acting on it must, in whatever units of force it may be measured, be proportional to ma. That is, if the number of units of force acting on the mass is changed, the product ma must be changed in the same ratio.

Dynamical Measure of Force.—We have now to fix upon the unit of force which is the most convenient measure of forces in dynamical questions.

Suppose that a force acting on m lbs. produces a foot-second units of acceleration. The number of units in the force, whatever the unit of force may be, is always *proportional to ma*. We may so choose our unit as to make the number of units in the force *equal to ma*. For since the numerical measure of the force always bears the same ratio to ma, if we arrange so that it shall be equal to ma for some particular values of m and a, it will always be equal to ma.

For this purpose we take the unit of force to be that force which produces in a pound mass an acceleration of one foot per second per second.

Then the number of units of force producing acceleration a feet per second per second in m pounds is ma.

The unit of force so chosen is called **the poundal.**

If, then, a force of $\mathbf{f}$ poundals acts on a mass of m lbs., and produces a foot-second units of acceleration, we have the relation

$$\mathbf{f}=ma.$$

This is the fundamental equation of Dynamics.

The force **one poundal** may be defined in several ways.

(1) **It is the force which gives to one pound a foot-second unit of acceleration.**

(2) **It is the force which, acting on a pound for one second, gives to it a velocity of one foot per second.**

(3) **It is the force which produces a foot-pound-second unit of momentum per second.**

With regard to the third method of defining the poundal, notice that the momentum produced by it in a second is always the same, on whatever mass it may act. If the mass is increased the velocity generated will become less, and *vice versa*. But the product of the mass in pounds and the velocity in feet per second, generated in one second, when a given force acts, is always the same; and if this force is one poundal, the product is unity. That is, the momentum generated in a second by a poundal in any mass whatever is a unit of momentum.

Absolute Unit of Force.—If, instead of using the foot, pound and second as the units of length, mass and time, any other units of these quantities be used, a corresponding unit of force would be obtained. Any such units of length, mass, and time which may be chosen arbitrarily are called *fundamental units.* The corresponding unit of force is called the *absolute unit of force* for the given system of fundamental units.

The absolute unit of force in any system of fundamental units may be defined as any of the three following:

(1) **The force which gives to unit of mass unit of acceleration.**

(2) **The force which gives to unit of mass unit of velocity per unit of time.**

(3) **The force which generates unit of momentum in unit of time.**

And in any system, if **f** absolute units of force produce a units of acceleration in m units of mass,

$$\mathbf{f}=ma.$$

In the metric system, in which the centimetre and the second are the units of length and of time, the gram mass is used as the unit of mass. This was defined as the mass of a cubic centimetre of pure water at the temperature of its maximum density, 4°C. The gram mass in actual use is, however, a little smaller than this, as has been found by more recent and exact observations.

The absolute unit of force in this system is called the **dyne.**

The dyne is the force which gives to a gram mass an acceleration of one foot per second per second.

Other definitions may be given, as in the case of the poundal.

These two systems are the only ones in practical use, and all scientific measurements are referred to the metric system. We shall now recapitulate what has been said about the systems of measurement, and state the connexions between the units of one system and the corresponding ones of the other.

Systems of Measurement and Connexions between them.—In the British system the fundamental units, or units of length, mass and time are the **foot, pound and second.**

This system is called the **Foot-Pound-Second, or F.P.S. system.**

In the Metric system the fundamental units, or units of length, mass and time are the **centimetre, gram and second.**

This system is called the **Centimetre-Gram-Second, or C.G.S. system.**

The absolute unit of force in the F.P.S. system is called the **poundal.**

The absolute unit of force in the C.G.S. system is called the **dyne.**

Fundamental Units.—A standard pound and a standard yard are kept in the Exchequer Office of London. The foot is defined as one-third of the standard yard.

A standard kilogram and a standard metre are kept in the Archives of Paris. The gram is defined as $\frac{1}{1000}$ of the standard kilogram, and the centimetre as $\frac{1}{100}$ of the standard metre.

The second is the fraction $\frac{1}{24 \times 60 \times 60}$ of the mean time of the earth's rotation on its axis relatively to the sun, or $\frac{1}{24 \times 60 \times 60}$ of the length of the mean solar day.

By comparison of these standards the following relations are found :

$$1 \text{ foot} = 30{\cdot}48 \text{ cms.,}$$
$$\text{or } 1 \text{ cm.} = {\cdot}03281 \text{ ft.}$$
$$1 \text{ lb.} = 453{\cdot}6 \text{ grams.}$$
$$1 \text{ gram} = {\cdot}0022046 \text{ lbs.}$$

EXAMPLE.—What force in poundals would give a ton mass an acceleration of 12 feet per second per second?

Let **f** poundals be the required force.

Then, by $\mathbf{f} = ma$,

$$\mathbf{f} = 2240 \times 12 = 26880.$$

Required force = **26880 poundals.**

EXAMPLE.—What velocity will a mass of 1 kilogram acquire in 10 seconds under the action of 1 dyne?

Let a cm.-sec. units be the acceleration.

Then, by $\mathbf{f} = ma$,

$$1 = 1000a,$$
$$a = \tfrac{1}{1000}.$$

$\therefore$ velocity acquired in 10 secs. = $\frac{1}{100}$ **cms. per sec.**

EXAMPLE.—Express a poundal in dynes.

Let 1 poundal be **f** dynes.

Now $\quad 1 \text{ lb.} = 453{\cdot}6$ grams,

and $\quad 1 \text{ ft.} = 30{\cdot}48$ cms.,

so that 1 ft. per sec. per sec. = 30·48 cms. per sec. per sec.

Then, ∵ 1 poundal gives to 1 lb. 1 ft.-sec. unit of acceleration, **f** dynes give 453·6 grams 30·48 cm.-sec. units of acceleration.

$$\therefore \quad \mathbf{f} = 453{\cdot}6 \times 30{\cdot}48 = 13826.$$

1 poundal = **13826 dynes.**

Exercises XIV. a.

1. What acceleration will a force of 4 poundals give to a mass of 100 lbs. ?

2. What force will give to a cwt. a velocity of a mile an hour in one second ?

3. What mass in grams must be acted on by a dyne to have an acceleration of a foot per second per second ?

4. What number of dynes will give to a gram an acceleration of 1 foot per second per second ?

5. What number of dynes will give to 10 lbs. an acceleration of 1 cm. per second per second ?

6. How many poundals will be required to give to 4 kilograms an acceleration of 3 foot-second units ?

7. How many poundals will give to a gram an acceleration of 1 cm. per second per second ?

8. What mass in grams must be acted on by a dyne to have an acceleration of a foot per second per second ?

In many questions the dynamical equation, $\mathbf{f} = ma$, and one (or more) of the kinematical equations, have to be used.

Whenever this is the case the acceleration has first to be found from one equation and the value thus found to be used in the other. The acceleration, in fact, may be called the connecting link between the dynamical and the kinematical part of the solution.

EXAMPLE.—A force of 40 dynes acts on a kilogram for one minute. What velocity does it generate?

Let a cm.-second units be the acceleration.

Then, by $$\mathbf{f}=ma,$$
$$40=1000\,a,$$
$$a=\tfrac{1}{25}.$$

The velocity generated is, by $v=u+at$, equal to $\frac{1}{25}\times 60$, or **$2\frac{2}{5}$ cms. per second.**

EXAMPLES.—A certain force acting on a ton through a distance of 5 feet changes its velocity from 3 to 7 feet per second. What is the force?

Let a foot-second units be the acceleration.

Then, by $$v^2=u^2+2as,$$
$$7^2=3^2+2\,.\,a\,.\,5\,;$$
$$a=4.$$

$\therefore$ the force is $2240 \cdot 4=$ **8960 poundals.**

EXAMPLE.—A carriage on rails, having at first a velocity of 30 feet per second, is observed in 20 seconds to travel 580 feet. The mass of the carriage is $4\frac{1}{2}$ tons. What is the force of resistance to motion?

Let a feet per second per second be the acceleration in the direction of motion.

Then, by $$s=ut+\tfrac{1}{2}at^2\,;$$
$$580=30\,.\,20+\tfrac{1}{2}a(20)^2\,;$$
$$200\,a=580-600=-20\,;$$
$$a=-\tfrac{1}{10}.$$

Thus the resistance causes an acceleration, in the direction in which it acts, of $\frac{1}{10}$ foot-second unit.

$\therefore$ the resistance is $4\frac{1}{2}\times 2240\times\frac{1}{10}$ poundals $=$ **1008 poundals.**

In the last example notice:

In the kinematical equations a is always measured in the same sense as s, u, and v; that is, this is the positive sense of a; and the sign of a indicates whether there is an acceleration or a retardation of the motion. In using these equations great care has to be taken about the signs of quantities, especially of a.

In the dynamical equation, $\mathbf{f}=ma$, there need be no trouble about sign. a must be of the same sign as $\mathbf{f}$ ($\because$ m is positive), and we may always consider them both to be positive. When we know in which sense the acceleration is, this denotes in which sense the force acts, and the numerical value of the acceleration, with that of the mass, gives the magnitude of the force.

Thus, in the above example the negative algebraical value of a indicates that the force is a retarding one, and the numerical values of a and m give the magnitude of the force.

Again, conversely, if we know the magnitude of the force and that of the mass we find the magnitude of a by simple division, and the algebraical sign to be given to a must then be determined by the nature of the question. This is exemplified in the following:

Example.—A half-ounce bullet is fired through a 3-inch board. The mean resistance to motion is 5000 poundals. If the bullet has a velocity of 1200 feet per second before striking the board and the resistance offered by the board is 5000 poundals, find the velocity of the bullet after passing through the board.

If a foot-second units is the acceleration in the board; then, by $\mathbf{f}=ma$, since $\frac{1}{32}$ lb. is mass of bullet.

$$5000=\tfrac{1}{32}\cdot a;$$
$$\therefore\ a=160000.$$

Let v feet per second be velocity after passing through board.

$$\therefore \text{ by } v^2=u^2+2as,$$
$$v^2=(1200)^2+2(-160000)\cdot\tfrac{1}{4}$$
$$=1440000-80000$$
$$=1360000.$$
$$\therefore\ v=1166.$$

Velocity required is **1166 feet per second.**

Exercises XIV. b.

1. For how long must a poundal act on a ton to give it a velocity of one foot per second?

2. What force acting on a mass of 7 kilograms gives it a velocity of 16 cms. per second while moving it from rest through 32 cms.?

3. A mass of 2 grams is moving with a velocity of 16 cms. per second, and is acted on by a retarding force of 5 dynes. How far will it go before coming to rest?

4. What velocity must be given to a mass of one hundredweight, acted on by a force of 14 poundals in the direction of its motion, that it may move through 9 feet in 4 seconds?

5. What velocity must be given to a mass of one hundredweight, acted on by a force of 14 poundals opposite to the direction of its motion, that it may move through 9 feet in 4 seconds.

6. What velocity in feet per second will a mass of 2 lbs. acquire if it is moved from rest through a distance of 3 yards by a force of 16 poundals?

7. For how long must a force of 8 dynes act on a mass of a kilogram to change its velocity from ·4 metres per second to ·6 metres per second in the opposite direction?

8. What time will a mass of 2 ounces, starting with a velocity of 6 feet per second and acted on by a force of 2 poundals in the direction of its motion, take to travel 44 feet?

Summary.

In **Dynamics** a force is estimated by its effect in producing motion in a body.

Every body resists the action of forces tending to set it in motion or to stop its motion, and the resistance is greater the heavier the body.

Inertia is the property of a body in virtue of which it resists the action of forces tending to change its state of motion.

Mass, which is sometimes called the quantity of matter in a body, is the same as its inertia.

If a body is set in motion under conditions in which the force opposing the motion is made very small, then the rate at which the velocity falls off is very small. And the smaller the opposing force is made the more nearly does the velocity become uniform.

These observations, and, more particularly, indirect observations on the heavenly bodies, lead us to infer the truth of

Newton's First Law of Motion.

Every body continues in a state of rest or of uniform motion in a straight line, except in so far as it is compelled to change that state by external impressed force.

If a constant force acts on a given mass it produces in it a constant acceleration.

To change the acceleration in any ratio, mass being constant, the force must be changed in the same ratio.

When the mass is changed in any ratio, the acceleration being constant, the force must be changed in the same ratio.

Therefore, force is proportional to the product mass × acceleration.

Momentum = mass × velocity,

It follows that: *force is proportional* to rate of change of momentum which it produces. This is expressed in

Newton's Second Law of Motion.

Rate of change of momentum is proportional to the acting force and takes place in the direction in which the force acts.

The force producing in mass m pounds an acceleration of a feet per second per second is *proportional to* ma however force is measured.

Take the force producing in mass 1 lb. acceleration 1 foot per sec. per sec. as one unit of force.

Then, force producing acceleration a feet per sec. per sec. in m pounds is *equal to* ma units.

This unit of force is called the **poundal.**

The momentum generated in one second in *any* mass by a poundal is one foot-pound-second unit of momentum.

In any system of units the *absolute unit of force* is the force which gives to unit of mass unit of acceleration.

If f units of force act on mass m and produce acceleration a, then $f = ma$.

This is the fundamental equation of Dynamics.

The absolute unit of force in the metric system, or the force which produces in one gram an acceleration one cm. per sec. per sec. is called the **dyne.**

CHAPTER XV.

DYNAMICAL MEASURE OF WEIGHT. ATTWOOD'S MACHINE.

Dynamical Measure of Weight.—The weight of a body is the force with which the earth attracts it to itself.

This force can be measured in dynamical units, like any other force, if we know what acceleration it can produce in a certain mass. Thus, if we know the mass of the body and the acceleration with which its weight moves it, that is, the acceleration with which it falls, if allowed to fall freely, we can determine the dynamical measure of its weight.

Consider, for instance, the weight of a pound. This is a force which may be specified as a certain number of poundals. Now the weight of a pound gives to a mass of a pound an acceleration of 32 foot-second units; for a pound (or any other mass) if allowed to fall freely falls with an acceleration of 32 units.

$\therefore$, by the equation $\mathbf{f}=ma$, the weight of a pound is $1 . 32$ units, that is, 32 poundals.

In general, suppose the weight of m pounds to be $\mathbf{w}$ poundals. Let g feet per second per second be the acceleration due to gravity.

Then since the force w poundals produces g foot-second units of acceleration in m pounds,

$$\mathbf{w}=mg.$$

This same relation must hold in any system of units whatever, $\mathbf{w}$ being the number of absolute units of force in the weight of m units of mass, and g being the measure of the acceleration due to gravity in the given units of mass and time.

In the F.P.S. system we have, *weight of a pound* = 32 *poundals.*

In the C.G.S. system, $g=981$. That is, the weight of a gram produces 981 cm.-sec. units of acceleration in a gram.

$\therefore$ *weight of a gram* $=981$ *dynes.*

The Weights of all Bodies are proportional to their Masses. —Experiment shows that at a given place the accelerations of all bodies due to their own weights are the same. Hence, from the formula

$$\mathbf{w}=mg,$$

it follows, since g is the same for all bodies, that, at a given place, the weights of all bodies are proportional to their masses.

This fact is made use of in weighing. A body is weighed in order to determine its mass. If we say that a body weighs a pound we mean that its weight is equal to that of a standard mass of a pound, or that the earth's attractive forces on them are equal. Hence we infer that their masses are also equal.

A pound of iron and a pound of wood are two masses which possess the same inertia, and will always possess the same inertia whether they are in the same or in different localities. As long as they are in the same locality they possess, as far as experiment can show, exactly the same weights; but the weight of either varies slightly as we move it about into various localities to which we have access.

The simplest way of showing, with a moderate degree of accuracy, and by means of an easy experiment, that, in a given locality, the weights of bodies are proportional to their masses, is to show that different bodies fall through equal distances in equal times. For instance a ball of iron and a ball of wood let fall from the same height, at the same instant, will reach the ground simultaneously.

A light body, such as a feather, is found to fall more slowly than a dense one such as a piece of lead. But this is due to the resistance of the air, which, being greater in comparison to the weight of the light body, produces more effect on it. If a special experiment is made in which the air is removed from the space in which two bodies, such as a feather and a piece of lead, fall, they will be found to fall through the same distance in the same time. Or, we may say, *in a vacuum all bodies fall with equal rapidity.*

Now by the formula $s=\frac{1}{2}at^2$, it follows that if two bodies move through the same distance s in the same time t, the acceleration is the same for both.

Thus the accelerations of the iron and of the wood due to gravity are equal.

Then, by the formula $\mathbf{f} = ma$, it follows that the forces producing these accelerations in the two masses are proportional to the masses. And these forces are the weights of the masses.

Let g be the common acceleration; m_1, m_2 the masses of the bodies; $\mathbf{w}_1$, $\mathbf{w}_2$ their weights.

Then
$$\mathbf{w}_1 = m_1 g,$$
$$\mathbf{w}_2 = m_2 g;$$
$$\therefore \frac{\mathbf{w}_1}{\mathbf{w}_2} = \frac{m_1}{m_2}.$$

EXAMPLE.—If a pound's weight acts on a ton, find the acceleration produced.

Mass moved = 2240 lbs.

Force = 32 poundals.

Let a feet per second per second be the acceleration.

Then, by
$$\mathbf{f} = ma,$$
$$32 = 2240a,$$
$$a = \tfrac{1}{70}.$$

The acceleration is $\frac{1}{70}$ **ft. per sec. per sec.**

EXAMPLE.—What force, in grams' weight, is required to give to 2 kilograms a velocity of 2 metres per second in $\frac{1}{4}$ of a second?

Mass moved = 2000 grams.

Vel. acquired in $\frac{1}{4}$ sec. is 200 cms. per sec.

$\therefore$ acceleration $= 200 \div \frac{1}{4} = 800$ cms. per sec. per. sec.

$\therefore$ required force $= 2000 \times 800$ dynes

$= 1600000$ dynes

$= \frac{1600000}{981}$ grams' wt.

= **1631 grams' weight nearly.**

EXAMPLE.—How many F.P.S. units of momentum will a ton's weight generate in a minute?

$$\text{Force acting} = 2240 \times 32 \text{ poundals.}$$
$$\therefore \text{ momentum generated per second} = 2240 \times 32 \text{ units.}$$
$$\therefore \text{ momentum generated in 1 minute} = 2240 \times 32 \times 60 \text{ units}$$
$$= \mathbf{4300800 \text{ units.}}$$

EXAMPLE.—A train has a velocity of 30 miles per hour, and the resistance to its motion is 8 lbs.' weight per ton of its mass. If left to itself, how far will it move before coming to rest?

Let the acceleration in the direction of the resisting force be a foot-second units.
Let the mass of the train be m tons.
Force is $m \times 8 \times 32$ poundals.
Mass is $m \times 2240$ pounds.

$$\therefore \; m \times 8 \times 32 = m \times 2240 \times a.$$
$$\therefore \; a = \tfrac{8 \times 32}{2240} = \tfrac{4}{35}.$$
$$\text{Initial velocity} = 44 \text{ feet per second.}$$

Let s feet be distance before coming to rest.

Then, by $$v^2 = u^2 + 2as,$$
$$0 = (44)^2 + 2(-\tfrac{4}{35})s,$$
$$s = \frac{(44)^2 \,.\, 35}{8} = 11 \,.\, 22 \,.\, 35 = 8470.$$

Required distance is **8470 feet.**

EXAMPLE.—A boy starts to slide and moves for 3 seconds before coming to rest, during which time he passes over 18 feet. Show that the resistance to motion is one-eighth of his weight.

Let a foot-second units be the acceleration in the direction of motion.

Then, by $$s = vt - \tfrac{1}{2}at^2,$$
$$18 = 0 - \tfrac{1}{2} \,.\, a \,.\, 3^2,$$
$$a = -4.$$

Let m lbs. be the boy's mass; and **f** poundals the resistance.

Then, by $\mathbf{f}=ma$,

$$\mathbf{f}=m\,.\,\dot{4}.$$

But boy's weight is 32 m poundals.

$\therefore$ resistance to motion $=\frac{1}{8}$ **of weight.**

Exercises XV. a.

1. A body is acted on by a force equal to one-third of its weight. Find its acceleration in foot-second units.

2. On what mass must a force of a kilogram's weight act to give it an acceleration of 2 metres per second per second?

3. If the weight of an ounce acts on a ton, what is the acceleration?

4. What force in lbs.' weight will give to 100 lbs. 40 foot-second units of acceleration?

5. A body moves from rest through 4 feet in 2 seconds. Show that it is acted on by a force equal to $\frac{1}{16}$ of its own weight.

6. A body's velocity is reduced from 16 to 12 feet per second in 4 yards under the action of a retarding force. What fraction of its weight is this force?

7. A train is brought to rest in 12 seconds in 120 yards. Show that the force of resistance is $\frac{5}{32}$ of its weight.

8. A train starts from rest and acquires a velocity of 40 miles per hour in half a minute. Show that the force exerted by the engine, in addition to that necessary to overcome the resistance to steady motion, is $\frac{11}{180}$ of the weight.

The motion of connected Bodies.—We shall now give some examples of the motion of bodies which are under the action of given forces, and are connected so that they must always have the same velocity, as, for instance, bodies which are connected by inextensible strings, or, again, bodies which always remain in contact with each other. In such cases it is generally the stress in the connexion which is required.

Consider the following as a typical example of this sort of problem.

EXAMPLE.—Two bodies of masses m and m' are connected by an inextensible string, and move in the line of the string under the action of a force **P** acting directly upon the mass m. Find the tension in the string.

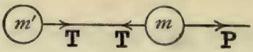

FIG. 140.—Motion of connected masses.

Since the string is inextensible, the two bodies must have the same acceleration.

First, find the acceleration of the system.

Since force **P** moves mass $m+m'$, acceleration is

$$\frac{\mathbf{P}}{m+m'}.$$

Now consider the motion of the mass m'.

Let the required tension in the string be **T**.

m' moves under the action of force **T**, and has acceleration

$$\frac{\mathbf{P}}{m+m'}.$$

Therefore, by formula $\mathbf{f}=ma$,

$$\mathbf{T}=\frac{m'\mathbf{P}}{m+m'}.$$

We may notice that the same result would be got, after finding the acceleration, by considering the motion of the mass m.

This moves under the action of the resultant force $\mathbf{P}-\mathbf{T}$.

$$\therefore\ \mathbf{P}-\mathbf{T}=m\,.\,\frac{\mathbf{P}}{m+m'},$$

$$\mathbf{T}=\frac{m'\mathbf{P}}{m+m'},\text{ as before.}$$

The form of the solution may be varied in the following way.

Apply the equation $\mathbf{f}=ma$ to the motion of each of the masses separately.

m moves under the action of $\mathbf{P}-\mathbf{T}$, and m' moves under the action of **T**.

Thus we have, if a is the acceleration,

$$\mathbf{P}-\mathbf{T}=ma,$$

$$\mathbf{T}=m'a.$$

As we do not require a we eliminate it from these two equations and obtain

$$m'\mathbf{P}-(m'+m)\,\mathbf{T}=0\,;$$

$$\therefore\ \ \mathbf{T}=\frac{m'\mathbf{P}}{m+m'}.$$

Notice that in solving this question to get our equation in **T**, we must consider the motion of one of the bodies separately. The equation thus got introduces another unknown quantity a. Therefore we must have another equation. This is most easily got by considering the motion of the whole system, as was done in the first cast.

EXAMPLE.--A body of mass m, hanging by a string, draws a body of mass m' along a smooth horizontal table, the string passing over a pulley at the edge of the table. Find the acceleration, the tension in the string, and the pressure of m' on the table.

The two masses move with a common acceleration under the action of certain forces. Although they move in different directions, their acceleration and their velocity at each instant will be the same as if they moved in the same straight line under the action of these forces; and we may consider the motion of the bodies as a whole as if this were the case.

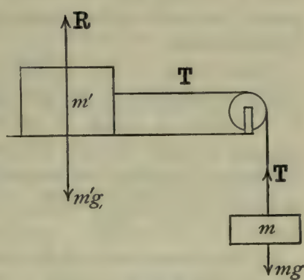

FIG. 141.

Let a be the acceleration,
T the tension in the string,
and **R** the upward pressure of the table against m'.
The bodies move under the action of the weight of m, that is, the force mg.

$$\therefore\ \ mg=(m+m')a.$$

$$\therefore\ \ a=\frac{mg}{m+m'}.$$

Consider the motion of m'.

This moves horizontally under the action of the force **T**.

$$\therefore \mathbf{T} = m'a$$

$$= \frac{mm'g}{m+m'}.$$

Since m' is acted upon by the two forces $m'g$ vertically downwards and **R** vertically upwards, and since it has no vertical motion,

$$\therefore \mathbf{R} = m'g.$$

Notice carefully this last conclusion. It is a result of the second law of motion which says that change of motion is in the direction of the acting force; or we may say the entire resultant force acting on the body must be in the direction of the change of motion. Now the entire force is in this case made up of **T** horizontally and **R** and $m'g$ vertically. Of these **T** is in the direction of the change of motion, or of a. It follows that **R** and $m'g$ must be equal in magnitude; for if they were not, the resultant force would have a vertical component as well as the component **T**, and, therefore, could not be horizontal.

Notice that in any question of this sort the tension of the string is assumed to be the same throughout. This is the case if no force is spent in setting the pulley in motion either against its own inertia or against frictional resistance, and if the string required no resultant force to set it in motion; that is, if we can neglect friction in the pulley and the masses of the pulley and the string.

In this question it might seem at first as if the tension of the string which has a mass m attached to its end is the weight of this mass. This would be so if the string held the mass at rest, or even if the mass moved down or up with uniform velocity. But it is clear that the weight must be in excess of the tension from the fact that the mass moves with downward acceleration, so that there must be a resultant downward force acting upon it. That is, the force pulling it down is greater than the force pulling it up, or the weight is greater than the tension.

EXPERIMENT 39.—Place a loaded trolley of entire mass m (say about 10) pounds on a flat, smooth, horizontal board placed on a table and raised up about 4 or 5 feet from the floor. Pass a strong fine string from it over a light smooth pulley at the end of the board, so that when the string between trolley and pulley is stretched it is horizontal. Let the end of the board with the pulley project beyond the edge of the table so that the rest of the string may hang vertical. Attach a light pan to this end of the string. Find the mass m' lbs. (including that of the pan) which must be carried by the string so as just not to move the trolley. The weight of this just balances the friction. Now suspend an additional mass m'' lbs. to the string, so as to make the trolley run along the board slowly. The weight of m'' lbs., or $m''g$ poundals, is the effective force causing motion in the whole mass $m+m'+m''$.

With the help of a clock beating seconds, or with a watch ticking at a known rate and held close to the ear, observe the distances passed over in 1, 2, 3, etc., seconds from starting. This may be done by releasing the trolley at a definite tick and noticing the distance passed over at the instant of the tick indicating the lapse of the time in question. For the distance passed over in any given number of seconds a single observation is not sufficient. With a little practice several estimations of the distance described in a given time can be made very close to each other. When this is done the mean of several observations, say four or five, should be taken for the distance required.

The distances may be measured by having a scale attached to the board, so that the trolley runs close by the side of it; or marks may be made on the board, and the distances measured.

Draw up a table of the distances for 1, 2, 3, etc., seconds. Since the force is constant and, therefore, the acceleration uniform, these distances should be proportional to 1^2, 2^2, 3^2, etc.

If the distances are not found proportional to the times it is probably because the board does not offer a uniform frictional resistance to the motion, or because the wheels or pulley do not work evenly. All this may be tested by seeing whether the same force is required just to start the trolley in various positions.

Deduce from each of the observations of the distances passed over in the various times, and by using the formula $s=\frac{1}{2}at^2$, values of the acceleration with which the trolley moves. These should be equal.

EXPERIMENT 40.—With the same apparatus keep the entire mass $m+m'+m''$, which is in motion, constant; but vary the moving weight. This may be done by taking weights out of the pan and putting them into the trolley. The friction force, measured by the weight of m, will vary extremely little with such small variations of the mass of the trolley, and, when there is very little friction it will generally be found that this force is practically the same throughout the experiments.

With each moving force determine the corresponding acceleration by letting the trolley run for a considerable time, say 3 seconds, if possible.

Tabulate the moving forces and the acceleration. These should be proportional to each other.

If a is the value of the acceleration in any one of the experiments, since the moving force is $m''g$ and the mass is $m+m'+m''$, therefore

$$m''g=(m+m'+m'')a;$$

$$g=\frac{(m+m'+m'')a}{m}.$$

Find the value of g from each of the experiments.

EXAMPLE.—Two masses m, m' are attached to the ends of a string which passes over a light pulley. m, which is the greater, descends, drawing up m'. Find the acceleration and the tension in the string.

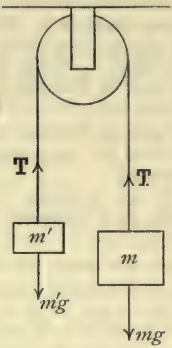

FIG. 142.

Let a be the acceleration, and **T** the tension in the string.

The forces acting on the system of two masses are the weight of m in the direction of the motion, and the weight of m' in the direction opposite to the motion.

Thus the motion takes place under the action of the force $mg-m'g$.

$$\therefore\ mg-m'g=(m+m')a.$$

$$a=\frac{(m-m')g}{m+m'}.$$

Consider the motion of m'.

This takes place under the action of the force $\mathbf{T}-m'g$.

$$\therefore\ \mathbf{T}-m'g=m'a.$$

$$\mathbf{T}=m'g+\frac{m'(m-m')g}{m+m'}=\frac{2mm'g}{m+m'}.$$

EXAMPLE.—In the last example a mass M is placed upon m. Find the pressure between them.

Let **P** be the required pressure, and a the acceleration.

Mass in motion is $m+m'+M$.

Moving force is $(m+M-m')g$.

$$\therefore \quad a=\frac{m+M-m'}{m+m'+M}g.$$

M moves under the action of the force $Mg-\mathbf{P}$.

$$\therefore \quad Mg-\mathbf{P}=Ma,$$

$$\mathbf{P}=M(g-a)$$

$$=Mg\left(1-\frac{m+M-m'}{m+m'+M}\right)$$

$$=\frac{2Mm'g}{m+m'+M}.$$

As a check we may find $\mathbf{P}$ another way.

Consider the motion of the system composed of m and m'. This is acted upon by the pressure $\mathbf{P}$ in the sense of the motion.

Entire force causing motion is $\mathbf{P}+(m-m')g$.

$$\therefore \quad \mathbf{P}+(m-m')g=(m+m')a=\frac{(m+m')(m+M-m')g}{m+m'+M}.$$

$$\mathbf{P}=\frac{2Mm'g}{m+m'+M}.$$

These results are not given that they should be remembered, but the examples have only been worked to indicate how any cases of the sort should be dealt with. In all cases they should be worked by applying the force formula, $\mathbf{f}=ma$, to the motion of the system and to the motion of such parts of the system as it may be necessary to consider.

Experiment 41.—Fix a light, very smoothly running pulley about 8 feet from the floor. Pass a fine string about 8 feet long over it; and attach scale pans to the ends of the string. Load the pans so that the entire mass on each side is m lbs. With a very light pulley and a very fine string a suitable load on each side is about 1 lb. or $\frac{1}{2}$ lb. Now a very small additional weight on one side should cause motion if the pulley is very good and the bearings well lubricated.

Place a vertical scale so that the pan on one side may run up and down close to it.

Add to this pan an additional load m' lbs. About $\frac{1}{2}$ oz. or 1 oz. may be tried, so that m' would be $\frac{1}{32}$ or $\frac{1}{16}$.

Then the mass in motion is $2m+m'$. And the force producing motion is $m'g$ poundals.

By holding the loaded pan high up and releasing it at a definite instant, notice, in successive trials, the distances passed over in 1, 2, 3, etc., seconds.

These should be proportional to the squares of the times, that is, to 1, 4, 9, etc.

From each of the observations deduce the value of the acceleration by the formula $s=\frac{1}{2}at^2$.

From the value of the acceleration find g.

> EXAMPLE.—Masses of 4 and 5 grams are fastened to the end of a string which passes over a light pulley, and a 2 gram mass is hung by another string below the 4 grams. The whole is free to move. Find in grams' weight the tension in the string connecting the 2 and 4 grams. If this string is cut when the 4 grams has descended 2 seconds, find how far it will descend altogether.

Let a cms. per sec. per sec. be the acceleration of the system before the string is cut.

Let $\mathbf{T}$ dynes be the tension.

Mass in motion = 11 grams.

Moving force $=g$ dynes.

$$\therefore \quad a=\frac{g}{11}.$$

The mass 2 grams moves under the action of force $2g-\mathbf{T}$ dynes.

$$\therefore \quad 2g-\mathbf{T}=2\,.\,a=\frac{2g}{11}.$$

$$\therefore \quad \mathbf{T}=\frac{20g}{11}.$$

Tension of string $=1\frac{9}{11}$ **grams' weight.**

Let a' cms. per sec. per sec. be the upward acceleration of the 4 grams after the string is cut.

Mass 9 grams is moving under action of g dynes.

$$\therefore \quad g=9a',$$

$$a'=\frac{g}{9}.$$

With acceleration $\frac{g}{11}$ the 4 gram mass descends in 2 seconds a distance

$$+\tfrac{1}{2}\frac{g}{11}\,.\,2^2 \text{ cms.} = \frac{2g}{11} \text{ cms.}$$

It then has downward velocity $\frac{2g}{11}$ cms. per sec.

With upward acceleration $\frac{g}{9}$ the 4 gram mass descends, before coming to rest, a distance

$$\left(\frac{2g}{11}\right)^2 \div \frac{2g}{9} \text{ cms.} = \frac{18g}{121} \text{ cms.}$$

$\therefore$ entire distance descended is

$$\frac{2g}{11} + \frac{18g}{121} \text{ cms.} = \frac{40g}{121} \text{ cms.} = \mathbf{324{\cdot}3\ cms.}$$

Notice the advantage of keeping the symbol g in the working, instead of substituting its numerical value, the cumbrous 981. To find the tension in grams' weight, the value of g is not required at all; and even to find the distance it is better to first find it in terms of g, and then substitute the numerical value of g.

EXAMPLE.—A body of mass 5 lbs. rests on an inclined plane of slope 2 in 15, and another body of mass 3 lbs. is attached to it by a string which passes over a pulley at the top of the plane. Find the acceleration of the motion and the tension in the string.

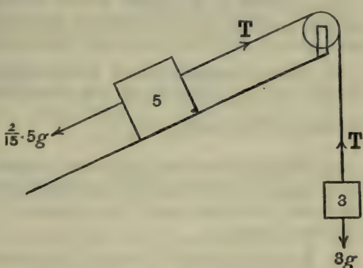

FIG. 143.

The system is acted on in the line of motion by the resolved part of the weight of 5 lbs. down the plane, and by the weight of 3 lbs. vertically downwards.

These forces are $\frac{2}{15}\,.\,5g$ and $3g$ poundals.

The second being the greater, the system will move so that the 3 lb. mass descends vertically, and the 5 lb. mass goes up the plane.

Force causing motion is

$$(3g - \tfrac{2}{15} \cdot 5g) \text{ poundals} = \frac{7g}{3} \text{ poundals.}$$

Let acceleration be a feet per second per second.

Then
$$\frac{7g}{3} = 8 \cdot a,$$

$$a = \frac{7g}{24} = 9\tfrac{1}{3}.$$

Let the tension in the string be $\mathbf{T}$ poundals.

By the motion of the 3 lb. mass,

$$3g - \mathbf{T} = 3a = \frac{7g}{8}.$$

$$\therefore \ \mathbf{T} = \frac{17g}{8}.$$

The acceleration is **$9\tfrac{1}{3}$ feet per second per second,** and the tension of the string is **$2\tfrac{1}{8}$ lbs.' weight.**

Exercises XV. b.

1. A force of 8 dynes draws a mass of 7 grams horizontally, which draws a mass of 2 grams behind it. What is the acceleration? and what is the tension in the string connecting the masses?

2. If two masses m and n are connected by a string and a force P acts on the mass m in the direction from n to m, show that the tension in the string is $\frac{nP}{m+n}$.

3. Three pound weights are connected by strings in a straight line, and are acted on by a force of a pound's weight in the same line. How far will they move in a second? and what are the tensions in the two strings?

4. A pound weight hanging by a thread draws a 5 pound weight along a smooth horizontal table. What is the tension in the thread? and in what time will the system travel 6 feet?

5. A 56 pound weight is placed at one edge of a perfectly smooth table 6 feet long, and a half ounce weight is attached to it by a string hanging over the opposite edge. In what time will the 56 pound weight travel to the opposite edge?

6. A 10 pound weight rests on a smooth table. A pound weight hangs by a thread which is attached to the 10 pound weight, and passes over a smooth light pulley at the edge of the table. A second pound weight is placed on the top of the first. Find the force of pressure between the two.

7. A 4 pound weight is on a smooth table. What mass must be attached to it by a string passing over the edge so that the two may move with acceleration $10\frac{2}{3}$ feet per second per second?

8. What mass must hang from a thread hanging over the edge of a smooth table and attached to a mass of 100 pounds on the table so that the whole may pass over 6 feet in a minute?

9. Masses of 2 and 3 pounds hang by a thread passing over a light pulley. Find the acceleration and the tension in the thread.

10. Two pound masses are attached to the ends of a thread passing over a pulley. If an ounce is removed from one and added to the other, through what distance will they pass in two seconds?

11. Masses m, m' at the ends of a string over a pulley move through a distance s and acquire a velocity v. Prove that

$$(m+m')v^2=2(m-m')gs.$$

12. Two masses, 10 and 12 ounces, are at the ends of a string passing over a light pulley. In what time will they move through 11 feet?

13. Two 250 gram weights are fastened to the ends of a string over a pulley. A 10 gram weight is placed upon one of them so that they begin to move. Find the pressure which the 10 gram weight exerts.

14. A string over a light pulley carries at its ends two scale pans each weighing 20 grams. Masses 40 and 50 grams are placed in the pans. Find the pressures which they exert.

15. Masses of 490 and 510 grains are attached to the end of a string over a light pulley, and when left free to move pass through 5 feet in 4 seconds. What value does this give for the acceleration due to gravity?

16. How could an experiment be arranged with a light smooth pulley to show the motion of 6 pounds under the actions of 2 pounds' weight?

17. How could you show the motion of 10 pounds under the action of the weight of half an ounce?

18. A 10 pound mass is on an inclined plain whose slope is 1 in 3. It is attached to a 5 pound mass hanging at the end of a string passing over a pulley at the top of the plane. Find the acceleration of the 10 pound mass and the tension of the string.

19. How long will 1 pound hanging vertically take to draw 4 pounds up 10 feet of an inclined plane of slope 1 in 5?

20. A 10 pound weight is placed on a smooth inclined plane of slope 1 in 5 and is connected by a string passing over a smooth pulley at the top of the plane to a pound weight which hangs by the string. Find the acceleration of the 10 pound weight and the tension of the string.

21. Two inclined planes of common height, 3 feet, and of lengths 5 and 6 feet, are placed back to back. Weights 10 and 11 pounds are placed on them and connected with a string passing over the top. Find the acceleration with which they move and the tension in the string.

22. An engine exerts a steady pull of half a ton's weight on a train of 40 tons. The resistance to motion is 15 pounds' weight per ton of the train. Find the acceleration and the pull on the last carriage, which weighs 4 tons.

Attwood's Machine.—Attwood's machine is an apparatus for illustrating the laws of motion and for investigating experimentally the acceleration due to gravity.

It must be noticed clearly that the experiments that can be made with such an apparatus as this are not *proofs* of the laws of motion; they only show that, when the errors which are incidental to the nature of the apparatus are eliminated as far as possible, the results obtained are very nearly what we should expect from the laws of motion. Also the value of the acceleration due to gravity can only be approximately determined by this means as compared with the accuracy which can be attained in other experiments.

The apparatus has for its main parts:

(1) A light pulley, running on very smooth bearings, over which can pass a light strong cord.

(2) Masses A and B, which are attached to the ends of the cord.

(3) Overweights which can be added to the weight A, the one shown at C being in the form of a bar.

(4) A vertical scale H by which the motion of the masses may be measured.

(5) Attachments to this vertical scale, namely, a platform Q for A to fall on, a ring P for removing C at any desired point of the motion (P and Q can be set at any required position on the scale), and an arrangement for releasing A from the top at a definite instant.

(6) A clock beating seconds audibly.

I. To show that under the action of no force velocity is uniform.

The masses A and B are made equal, so that the system A, B would move under the action of no moving force. An overweight C is placed on A. The ring P is adjusted so as to remove C at a beat of the clock. This may be done by placing P so that C is removed just one second after A is released, A being released at a beat of the clock.

Various positions are then found for Q such that A reaches Q in 1, 2, 3, ... seconds after C is removed by P. It is found that the distances which A passes over between P and Q in these times are proportional to 1, 2, 3, ... Hence the velocity of the system with no moving force is uniform.

II. To show that the acceleration produced by a given overweight acting on given masses is constant.

This is done by showing that the velocity generated is proportional to the time for which it acts.

A and B are made equal.

Various positions are found for P, so that it removes C in 1, 2, 3, ... seconds after A is released.

Corresponding positions are

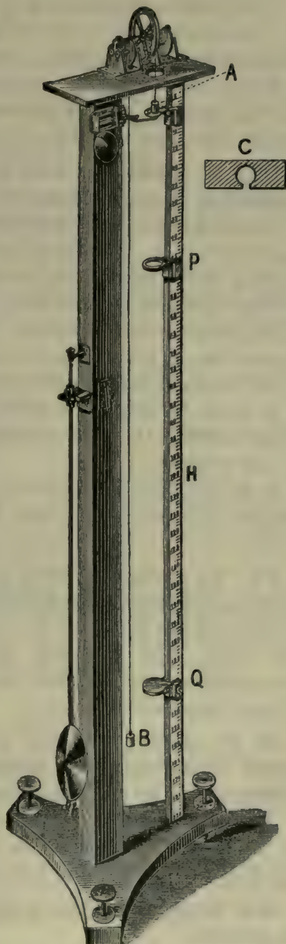

FIG. 144.—Attwood's machine.

found for Q, so that A reaches it always one second after C is removed by P.

The distances travelled by A from P to Q measure the velocities of A on reaching P in the various cases; and these are found to be proportional to 1, 2, 3,... seconds.

Hence the velocities generated by the weight of C in the moving system are proportional to the times for which this weight acts, that is, the acceleration is uniform.

III. To show that with a given mass in motion the acceleration is proportional to the overweight.

The entire mass, that is, the sum of A, B and C, is kept constant, and C is varied. (This may be done by removing equal amounts from A and B and adding the quantities removed to C.)

The velocities generated by C in one second, in the various cases, are measured as before; that is, by placing P to remove C in one second after A is released and by placing Q to receive A in one second after C is removed.

It must be noticed that in all cases the mass moved by the weight of C is the sum of the masses of A, B and C and not of A and B alone.

It is found that the velocities generated in one second by the weight of C are proportional to the mass and thus to the weight of C.

IV. To prove that with a given overweight the accelerations are inversely proportional to the mass moved.

A and B are increased by equal amounts each time, keeping C the same. The velocities generated by C in one second are measured as before. These are found to be inversely as the sums of the masses of A, B and C.

V. To show that with a given mass in motion and a given overweight the distance passed over from rest is proportional to the square of the time.

The ring P is removed, and positions found for Q to receive A 1, 2, 3, ... seconds after its release. The distances passed over by H are found to be as 1, 4, 9, ...

VI. To find the acceleration of a freely falling body due to gravity.

Let A and B be each of mass M, and C of mass m. If g is the acceleration of gravity in the units used, the acceleration of the moving system is

$$\frac{mg}{2M+m}.$$

This is measured in the way already described in II. Let it be a.

Then $$\frac{mg}{2M+m}=a.$$

$$\therefore\ g=\frac{a(2M+m)}{m}.$$

Necessary Corrections.—There are several causes of error in experiments made with Attwood's machine.

The wheel, although very light, requires some effort to set its mass in motion. This makes the velocity at any instant of the moving system, and its acceleration, less than the theoretical value.

The friction of the axle on the bearings has to be overcome, and this further reduces the velocity of the motion.

The air offers a slight resistance to the masses moving through it.

The mass of the string produces a slight effect, as it has to be set in motion; and when there is more of it on one side than on the other, the difference of weights of the two parts is a force concerned in the motion, helping or hindering it.

The method of measuring the times is a very rough one, and, as the times to be measured are all very short, only a few seconds, the errors made in the measurements may be considerable as compared with the quantities to be measured.

Exercises XV. c.

[The First Six Questions are on Attwood's Machine.]

1. If each of the masses is 1 lb. find the overweight in ounces in order that a velocity of one foot per second may be acquired in one second.

2. With an overweight of 10 grams, what must the mass on each side be that the acceleration may be ·5 metre per second per second?

3. What is the ratio of overweight to each mass to cause acceleration $\frac{g}{2}$?

4. What is the ratio of overweight to each mass to cause acceleration $\frac{g}{7}$?

5. If the entire mass moved, including the overweight, is two pounds, what must the overweight be and what must each mass be to cause an acceleration of 1 foot per second per second?

6. If the overweight is $\frac{1}{100}$ of each mass, in what time will the system move through 2 metres?

7, A mass of 5 lbs. is resting on a smooth horizontal plane. How many poundals must there be (i) in the horizontal force which, acting on the mass, will give it a velocity of 8 feet per second in $\frac{1}{3}$ of a second; and (ii) the vertical force which will cause the mass to ascend with an acceleration of 7 ft. per second per second. (Coll. Precep., 1898.)

8. Horizontal forces, equal respectively to a 3 pound weight and a 4 pound weight, act on a mass of 6 lb. placed on a smooth horizontal table, the directions of the forces making with each other an angle of 60°. Prove that the mass will move with an acceleration greater than that of a falling body (Coll. Precep., 1898.)

9. A train of mass 100 tons starts from rest. What force must the engine exert if the train is to acquire a speed of 15 miles per hour in the course of half a minute, the rate of increase being uniform? The total resistance is 10 lbs.' wt. per ton, the lines being level and straight. (Coll. Precep., 1897.)

10. A particle, of mass 2 lbs., on a smooth plane inclined at 60°, is drawn up by a particle, of mass 6 lbs., which descends vertically. The two particles are connected by a fine inextensible string passing over a small smooth peg at the top of the plane. Find the tension of the string and the acceleration of the particles. (Coll. Precep., 1897.)

11. A force equal to the weight of 2 lbs., acting on a certain mass, moves it from rest over 100 feet in 5 seconds. What is the mass? (Coll. Precep., 1897.)

12. Two particles, whose masses are P and Q, are connected by a fine thread; Q is placed on a horizontal table, and P hangs over the edge; there is no friction; if P is allowed to fall, find the acceleration.

If P is 1 lb. and Q is 7 lbs., find how long it would take P to fall through the first 3 feet of its descent. (Science and Art, 1897.)

13. Two particles, whose masses are 15 and 9 lbs., are connected by a fine thread which passes over a smooth point (as in Attwood's machine); if they are allowed to move, find the acceleration of the velocity and the tension of the thread. (Science and Art, 1899.)

14. A particle, whose mass is 4 grams, is acted on by two forces —4 dynes due north and 3 dynes due east. What is the magnitude of the resultant acceleration produced in it? (Oxford Locals, 1899.)

15. Suppose a mass of 10 tons moving at 880 feet per minute to be acted on for 5 minutes by a force, acting in the direction of motion, such that during the time of its operation the speed is changed to one of 8 miles per hour: find the mean values of the acceleration and of the force. State in what units your answers are expressed. (London Matric., 1899.)

16. What is the British absolute unit of force? Express in terms of this unit a force equal to the weight of 1 lb.

A force equal to the weight of $\frac{1}{2}$ oz. acts upon a mass of 2 lbs., find the velocity of the mass after it has traversed 4 feet. (Oxford Locals, 1897.)

17. Describe some way of causing a mass of 1 lb. to be acted on by a force equal to one-sixteenth of its weight, and find the velocity acquired by the body under such circumstances when it has moved from rest through a distance of one foot. (Camb. Jr. Loc., Stat. Dyn. and Hydro., 1897.)

18. A carriage weighing 1 lb. is placed on a smooth horizontal table and has attached to it a string which passes over a pulley fixed to the edge of the table. A weight of 1 oz. is tied to the end of the string. If the carriage is held a yard from the edge of the table and then let go, find how long it will take in reaching the edge. (Camb. Sr. Loc., Stat. Dyn. and Hydro., 1897.)

19. A force equal to the weight of 10 lbs. acts on a body for 5 seconds. By how much will the momentum of the body be changed during this time? (Camb. Sr. Loc., Stat. Dyn. and Hydro., 1898.)

20. Two masses of weights 2 and $2\frac{1}{2}$ lbs. are connected by a light string which passes over a pulley. Find the acceleration with which the larger weight descends, and the space through which it moves in the first 5 seconds of its motion. (Camb. Jr. Loc., Mech., 1898.)

Summary.

The weight of a body, or the force with which the earth attracts it, can be determined in dynamical measure if we know the mass of the body and the acceleration which its weight gives to it, or with which it falls freely.

In any system of units, if m is mass, g the acceleration due to gravity, and $\mathbf{w}$ the weight,

$$\mathbf{w} = mg.$$

Thus the weight of a pound is 32 poundals, and the weight of a gram is 981 dynes.

All bodies fall with equal rapidity: therefore *all bodies have weights proportional to their masses.*

Attwood's Machine is an apparatus for illustrating the laws of motion and for determining the acceleration due to gravity.

It consists mainly of a light smooth pulley over which passes a string carrying weights at its ends. The magnitudes of these weights can be adjusted. There are also fittings to the machine for checking the motion of the weights at any instant.

The results obtained by Attwood's machine are only to be regarded as approximate. The laws of motion are much more exactly (though indirectly) demonstrated by observations on the heavenly bodies. And the value of g is much more exactly found by means of other experiments.

CHAPTER XVI.

IMPULSE. NEWTON'S THIRD LAW OF MOTION.

Momentum produced by a Force.—Suppose that a force of $\mathbf{f}$ units acts on a body of mass m and causes in it an acceleration of a units. Then we have the relation

$$\mathbf{f} = ma.$$

Let t be the time for which the action takes place. From the above $$\mathbf{f}t = mat.$$

If v is the velocity produced in m by the action of $\mathbf{f}$, $v = at$.

Hence $$\mathbf{f}t = mv.$$

Or, *the momentum produced by the force* is equal to $\mathbf{f}t$.

We see that the momentum produced by the force $\mathbf{f}$ acting for a time t is the same, whatever be the mass on which it acts.

Suppose the force not to remain constant during the time for which it acts. Imagine the time to be divided into a number of intervals t_1, t_2, t_3, etc., during each one of which the force may be supposed to retain a constant value. Let the values of the force in these intervals be $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$, etc. Then, if the accelerations of m during the intervals are a_1, a_2, a_3, etc.,

$$\mathbf{f}_1 = ma_1,$$

$$\mathbf{f}_1 t_1 = ma_1 t_1,$$

and so for the others ;

$$\therefore \ \mathbf{f}_1 t_1 + \mathbf{f}_2 t_2 + \mathbf{f}_3 t_3 + \ldots = m(a_1 t_1 + a_2 t_2 + a_3 t_3 + \ldots).$$

But $a_1 t_1 + a_2 t_2 + \ldots$ is the entire velocity added, v, and if the mean value or time average of the force is $\mathbf{f}$,

$$\mathbf{f}_1 t_1 + \mathbf{f}_2 t_2 + \ldots = \mathbf{f}t\ ;$$

$$\therefore \ \mathbf{f}t = mv,$$

the same equation as before.

It should be noticed that, in these results, v is the velocity and mv the momentum *communicated to the body* by the action of the force **f**. And the results are true whether the body has a velocity before the action of **f** or not. We may call v the added velocity in the direction of **f**.

Suppose, however, that the body has an initial velocity u, and that v is its final velocity, both measured in the direction in which **f** acts, then

$$\because\ \mathbf{f}=ma, \text{ so that } \mathbf{f}t=mat, \text{ and } v=u+at, \text{ so that } at=v-u\,;$$

$$\therefore\ \mathbf{f}t=m(v-u).$$

This equation is frequently of use in solving questions about the action of a force and the time for which it acts. The questions to which it is applicable could also be solved by means of the fundamental formula of Dynamics, $\mathbf{f}=ma$, combined with the kinematical equations; but the work may sometimes be much shortened by its use.

The equation may be called *the momentum equation.*

Example.—For what time must a force of 2 lbs.' wt. act on a body of mass $1\frac{1}{2}$ tons to give it a velocity of 12 feet per second?

Let t seconds be the time.

The force is 64 poundals.

Then, by $\mathbf{f}t=mv$,

$$64t=1\tfrac{1}{2}\times 2240\times 12\,;$$

$$t=\frac{3\times 2240\times 12}{2\times 64}=630.$$

Required time is **630 secs., or 10 mins. 30 secs.**

Momentum is susceptible of Sign.—In using the complete momentum formula, $\mathbf{f}t=m(v-u)$, the signs of the quanities are likely sometimes to prove troublesome, and great care must be taken about them.

In many cases it will be sufficient to consider the entire change of momentum, and to notice that this is in the sense of the force, and equal to the product of force and time.

It should be noticed that momentum, like velocity, is a quantity susceptible of sign, so that to destroy momentum which exists in one sense is equivalent to communicating an equal amount in the opposite sense.

The various cases which can occur may be stated in greater detail as follows :

(i) If the force acts in the sense of the initial velocity, or, if this is zero, force × time is equal to the amount of momentum produced.

(ii) If the force acts in the sense opposite to the initial velocity, and reduces it or makes it zero, force × time is equal to the amount of previously existing momentum destroyed.

(iii) If the force acts in the sense opposite to the initial velocity and reverses it (that is, ultimately causes the body to move in the sense of the force), force × time is equal to the sum of the previously existing momentum destroyed and the opposite momentum produced.

All these cases are covered by the single statement

force × time = momentum produced in sense of force,

or by its algebraical equivalent, the momentum equation.

EXAMPLE.—A body of mass 5 lbs., moving with a velocity of 10 feet per second, meets with a resistance which acts on it for half a second and reduces its velocity to 8 feet per second. What is the resistance?

Body's loss of momentum, or momentum communicated in direction of force, $=5(10-8)=10$ units.

Let force be **f** poundals.

Then $$\mathbf{f}\cdot\tfrac{1}{2}=10.$$

$$\therefore\ \mathbf{f}=20.$$

The force is **20 poundals.**

In other cases it will be necessary to use the equation and to have a clear understanding about the signs of all the quantities involved. In such cases the following is probably the best plan.

Always consider **f** positive (as we have done in using the equation $\mathbf{f}=ma$); and reckon u and v positive if in the sense of **f**, negative if in the opposite sense. m and t are, of course, always positive.

The above example may be used to illustrate this, but the solution will be less simple than that already given.

Let the force be **f** poundals.
Use the formula $\mathbf{f}t = m(v-u)$.
Here $u = -10$, $v = -8$, both velocities being opposite to the direction in which **f** acts.

$$\therefore\ \mathbf{f} \cdot \tfrac{1}{2} = 5\{-8-(-10)\}$$
$$= 5(-8+10) = 10.$$
$$\therefore\ \mathbf{f} = 20.$$

The force is 20 poundals.

In the two following examples the full formula *must* be used, due regard being had to the signs.

Example.—A body of mass 5 lbs., moving with a velocity of 10 feet per second, is acted on by a force of 40 lbs.' wt. opposite to the direction of its motion for half a second. What is the velocity of the body at the end of this time?

Let the required velocity be v feet per second in the direction in which the force acts.
The original velocity in the same direction is -10 feet per second.

$$\therefore \text{ by } \mathbf{f}t = m(v-u),$$
$$40 \cdot 32 \cdot \tfrac{1}{2} = 5(v+10),$$
$$5(v+10) = 640,$$
$$v+10 = 128,$$
$$v = 118.$$

The required velocity is **118 feet per second in the sense opposite to the original velocity.**

Example.—A body of mass 5 lbs., moving with a velocity of 10 feet per second, meets with a resistance of 3 lbs.' wt., which acts on it for half a second. What is the velocity of the body at the end of this time?

Let the required velocity be v feet per second in the direction of the force.

Then by the formula $\mathbf{f}t = m(v-u)$,

$$3 \,.\, 32 \,.\, \tfrac{1}{2} = 5(v+10),$$
$$5(v+10) = 48,$$
$$v+10 = 9{\cdot}6,$$
$$v = -{\cdot}4.$$

The required velocity is **·4 feet per second in the direction in which it was first moving.**

In any case of this sort the result may be checked by considering that the momentum communicated must be in the direction of the force and equal to the product $\mathbf{f}t$.

Thus in the last example but one :

$$\text{Mom. communicated} = 5(118+10) = 640.$$
$$\text{Product} \quad \mathbf{f}t = 40 \,.\, 32 \,.\, \tfrac{1}{2} = 640.$$

In the last example,

$$\text{Mom. communicated} = 5(10-{\cdot}4) = 48.$$
$$\mathbf{f}t = 3 \,.\, 32 \,.\, \tfrac{1}{2} = 48.$$

Example.—A mass of 5 lbs. falls from a height of 9 feet on the ground and is brought to rest in $\frac{1}{10}$ of a second. What is the mean resistance of the ground?

The velocity of the mass just before striking the ground is $\sqrt{32 \,.\, 18}$ ft. per sec. $= 24$ ft. per sec.

Let the required resistance be $\mathbf{f}$ poundals.

The resultant upward force acting on the body while it comes to rest is

$$\mathbf{f} - 5 \,.\, 32 \text{ poundals.}$$

This force acts for $\frac{1}{10}$ sec. and destroys downward momentum $5 \,.\, 24$ units.

$$\therefore \quad (\mathbf{f} - 5 \,.\, 32)\tfrac{1}{10} = 5 \,.\, 24,$$
$$\mathbf{f} = 5 \,.\, 32 + 1200$$
$$= 1360.$$

Required resistance is **1360 poundals or 42½ lbs.' weight.**

[By the conditions of this question it is clear that the weight of the body is a force concerned in its motion, while it comes to rest, and therefore it has been taken into account.]

Exercises XVI. a.

1. What force must act on a body of 20 pounds mass to give it a velocity of 20 feet per second in 2 seconds?

2. For how long must a force of 1 pound's weight act on a hundredweight to increase its velocity by 5 feet per second?

3. What mass will have its velocity changed from 5 feet per second to 8 feet per second in the opposite direction by means of a force of 112 pounds' weight acting for 26 seconds?

4. An ounce mass moving with a velocity of 50 feet per second is brought to rest by $2\frac{1}{2}$ tons' weight. In what time is this done?

5. A half pound ball is struck so that it begins to move with a velocity of 120 feet per second. What is the magnitude of the force acting on the ball if it lasts for $\frac{1}{2}$ second? and what if it lasts for $\frac{1}{100}$ second?

6. A body of mass 2 tons is acted on by a force of 7 pounds' weight for 20 seconds and is then moving at 7 feet per second in the direction of the force. How was it moving at first?

7. A piece of wood of mass 2 pounds falls and reaches the surface of water with a velocity of 20 feet per second. It sinks through the water for $1\frac{1}{2}$ seconds. What is the mean upward pressure of the water against it?

8. A ball weighing 5 ounces is thrown up against the ceiling, which it reaches with a velocity of 5 feet per second, and rebounds with a velocity of 3 feet per second. If the ball is in contact with the ceiling $\frac{1}{12}$ of a second, with what average pressure in ounces' weight does the ceiling press it down?

Impulse.—If a very large force acts for a very short time, as, for instance, in the case of a cricket bat driving a ball, the force is called an **impulsive force.** Such a force is not of a different nature from the other forces which we consider in Mechanics; it only differs from these in magnitude and in the time for which it acts.

In the case of an impulsive force, it is generally difficult or impossible to estimate the value of the force or the time for which it acts; and indeed, as a rule, the force varies considerably during its action. It is, however, of more importance to know the effect produced by the force, or the momentum generated by it. This is called the **impulse** of the force; and the term is sometimes used in connexion with other forces besides those which are impulsive.

The impulse of a force is the product of the mean value of the force and the time for which it acts: this is also equal to the momentum generated by the force.

The word *blow* is also used sometimes to mean the measure of an *impulse.*

Note that *a force* is measured by *the rate at which it generates momentum* or *the momentum which it generates per unit of time; an impulse* is measured by *the entire momentum which it generates.*

Exercises XVI. b.

1. What impulse will cause a 2 pound mass to move with a velocity of 15 feet per second?

2. What is the magnitude of the impulse which brings a mass of $2\frac{1}{2}$ kilos., moving with a velocity of 12 metres per second, to rest?

3. A 4 pound ball is struck so that it moves off with a velocity of 20 feet per second. What is the mean value of the force if it acts for $\frac{1}{60}$ second?

4. A ball weighing half a pound moving at 25 feet per second strikes a wall and comes to rest. What is the impulse on the ball?

5. If the same ball rebound from the wall with a velocity of 10 feet per second instead of coming to rest, what is the blow on it from the wall?

Reaction between two Bodies.—If a body A presses on another B, B presses against A in exactly the opposite direction and with a force exactly equal to that which A exerts against B. If A acts on B in any manner B reacts on A with an equal force in the opposite direction.

This principle we have already employed in Statics, and in certain questions in Dynamics, about bodies connected by strings. It is universally true, whether the bodies A and B are at rest or in motion. It is contained in the following:

Newton's Third Law of Motion.—To every action there is an equal and contrary reaction.

It is, perhaps, not easy to realize that this is always true, in cases of motion as well as in cases of rest. It may seem, at first, natural to suppose that, if a body A begins to move in a straight line, drawing a body B after it, the force which A exerts on B is greater than that which B exerts on A to keep it back. But, in fact, these two forces are exactly equal, and there must be some force acting on A which is greater than that which A exerts on B by just the amount which is required to set A in motion in the given manner.

If a horse draws a cart the force which he exerts on the cart is equal to that with which the cart pulls him back. If an engine draws a train it exerts a force which is just equalled by that with which the train pulls it back. But what we may call the effective force acting on the engine, due to the steam pressure, is greater than the force which it exerts on the train, or which the train exerts on it by just the amount which is necessary to move the engine in the way in which it is moving.

The matter may be further illustrated by the following example. Suppose that an engine is drawing two identical carriages A and B. Suppose that A pulls B with a force $\mathbf{f}$ units. B pulls A back with a force $\mathbf{f}$ units. But a resultant force $\mathbf{f}$ units is required for the motion of A. Hence the force with which the engine acts on A is $2\ \mathbf{f}$ units. Of this $2\ \mathbf{f}$ units of force applied directly to A we may consider that $\mathbf{f}$ units are employed to produce its motion and the other $\mathbf{f}$ units go to produce the motion of B.

Suppose two bodies to act on each other with a given force of action between them for a given time, and so to set each other in motion. Then, since the forces acting on the bodies are equal and the times for which they act are equal, and since the momentum produced by a force is equal to the product of force and time, it follows that the momenta generated in the two bodies in opposite directions are equal to each other in magnitude.

This may be expressed in symbols as follows.

Suppose two bodies of masses m, M to act on each other with a force $\mathbf{f}$ for a time t.

Let v, V be the velocities generated in the two bodies (these being measured in opposite senses).

Then, since $\mathbf{f}t = mv$, and $\mathbf{f}t = MV$, $\quad \therefore\ mv = MV$.

Again we may suppose that the force of action between the two bodies does not remain constant during the time of action,

If the time average of the force is f, we still have,

$$\mathbf{f}t = mv, \text{ and } \mathbf{f}t = MV.$$

$$\therefore\ mv = MV.$$

Further, it is not necessary that the bodies should have been originally at rest; v and V may be regarded as the velocities communicated to the bodies by their inter-action, whether they had other velocities before or no. In this case mv, MV are not the final momenta of the bodies, but the momenta due to the action.

The result may be put in words as follows :

Any inter-action between two bodies generates in them equal momenta in opposite directions.

Of course it is supposed that the bodies are not acted upon by any other forces than those due to the action between them, because then in considering the momentum given to either body we should have to consider all the forces acting on it. If, for instance, one of the bodies is rigidly fixed, no momentum can be generated in it, as the fixings will always supply just enough force to prevent this.

The principle here given is one of very great importance in Mechanics, and examples of it frequently occur.

If a gun, free to move on a horizontal plane, discharges a shot, the momenta with which the gun and the shot move off are equal.

As an example of an inter-action of the nature of a pull, if two carriages on a perfectly smooth horizontal plane and some distance apart exert a pull on each other by means of a rope, they will then be moving towards each other with equal momenta.

The principle may also be stated in another way which is more convenient for some purposes. Since momentum is a quantity which, like velocity, has direction, and may therefore be regarded as having positive or negative sign, according to the sense of the velocity involved in it, we must consider two momenta in opposite directions along a straight line to have opposite algebraical signs. Hence, the algebraical sum of two equal and opposite momenta along a straight line is zero. And the two momenta caused by any inter-action between two bodies have a zero algebraical sum.

We may state this as follows :

The entire change in the joint momentum of two bodies due to any inter-action between them is zero.

EXAMPLE.—A gun weighing 80 tons discharges a shot of 100 lbs. with a velocity of 600 feet per second. What is the velocity with which the gun recoils?

Let V feet per second be the required velocity.

Then since momenta of gun and shot are equal, we have

$$80 \times 2240 \times V = 100 \times 600;$$

$$V = \frac{600}{8 \times 224} = \frac{75}{224}.$$

Required velocity is $\frac{75}{224}$ **feet per second.**

EXAMPLE.—A ball of mass 2 lbs., moving with a velocity of 7 feet per sec. strikes against another of mass 6 lbs., which is originally at rest, and rebounds with a velocity of 2 feet per sec. With what velocity does the second ball go on?

The first ball has at first $2 \cdot 7$ units of momentum.

After the blow it has $2 \cdot 2$ units of momentum in the opposite direction.

$\therefore$ the blow has given to it $2 \cdot 9$ units of momentum.

$\therefore$ the second ball also acquires $2 \cdot 9$ units of momentum.

If the required velocity of the second ball is v ft. per sec.,

$$6v = 2 \cdot 9.$$

$$\therefore \quad v = 3.$$

Required velocity is **3 ft. per sec.**

Alternative method: this question might be solved by considering that the entire momentum in the line of the motion of the system, consisting of the two balls, remains unchanged.

Let v feet per second be the required velocity.

Then, since the entire momentum in the line in which the balls move is the same after as before the impact,

$$6v - 2 \cdot 2 = 2 \cdot 7.$$

$$\therefore \quad v = 3.$$

EXAMPLE.—A ball of mass 2 lbs., moving with a velocity of 7 feet per second, strikes against a ball of mass 6 lbs. at first at rest, and the two go on moving together. What is their common velocity?

Let v feet per second be the required velocity.

Then, equating momenta before and after impact,

$$(2+6)v=2\,.\,7.$$
$$v=1\tfrac{3}{4}.$$

Velocity is **$1\frac{3}{4}$ ft. per sec.**

EXAMPLE.—Two balls of masses 6 and 2 move towards each other with velocities 4 and 5. They collide directly, and continue to move in the straight line in which they were first moving; and their relative velocity of separation after the impact is one half of their velocity of approach before impact. Find the velocities after impact, and the measure of the impulse.

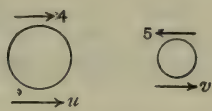

FIG. 145.

Let the required velocities be w and v, measured in the sense in which the ball 6 was first moving.

Denote the balls and their velocities as in the figure.

$$\text{Velocity of approach} =4+5=9.$$
$$\text{Velocity of separation}=v-u.$$
$$\therefore\ v-u=\tfrac{1}{2}\,.\,9=4\tfrac{1}{2}. \quad\text{........................(1)}$$

Equating momenta before and after impact,

$$6u+2v=6\,.\,4-2\,.\,5=14,$$
$$3u+\ v=7. \quad\text{..(2)}$$

From equations (1) and (2),

$$4u=2\tfrac{1}{2},$$
$$u=\tfrac{5}{8};$$
$$v=4\tfrac{1}{2}+\tfrac{5}{8}=5\tfrac{1}{8}.$$

The required velocities are $\frac{5}{8}$ **and $5\frac{1}{8}$ units, both in the sense in which the ball 6 was first moving.**

The impulse gives the ball of mass 6 a velocity of

$$4-\tfrac{5}{8}=3\tfrac{3}{8}\text{ units.}$$

$\therefore$ measure of impulse is $6\,.\,3\frac{3}{8}=$ **$20\frac{1}{4}$ F.P.S. units.**

[Or, since impulse gives to mass 2 a velocity of $5+5\frac{1}{8}=10\frac{1}{8}$ units, its measure is $2\,.\,10\frac{1}{8}=20\frac{1}{4}$ units, as before.]

EXAMPLE.—A body of mass 10 grams moving at 4 cms. per sec. strikes a body of mass 15 grams, and the two go on moving together. What force is necessary to bring them to rest in 3 secs.?

Combined momentum of the bodies

=original momentum of the 10 grams

$=10 \,.\, 4$ units.

Let **f** dynes be the required force.

Then $\mathbf{f} \,.\, 3 = 10 \,.\, 4.$

$\mathbf{f} = 13\frac{1}{3}.$

Required force $= \mathbf{13\frac{1}{3}}$ **dynes.**

Notice that the given number 15 does not come into the solution. The two bodies would go on with the original momentum of the first whatever be the mass of the other.

Exercises XVI. c.

1. A gun of mass 8 lbs. discharges a $\frac{1}{2}$-oz. bullet with a velocity of 1200 feet per second. With what velocity does the gun recoil?

2. An ounce bullet is fired into a block of wood weighing 36 lbs. and lodges in it; the wood begins to move off with a velocity of 2 feet per second. What was the velocity of the bullet?

3. A kilogram mass moving with a velocity of 10 metres per second overtakes and strikes a mass of 50 grams moving with a velocity of 5 metres per second. If after the impact the 50 grams mass moves with velocity 10 metres per second, what happens to the kilogram? And what is the measure of the blow?

4. A 7-lb. weight moving at the rate of 10 feet per second meets and strikes a cwt. moving 2 feet per second, and rebounds with a velocity of 4 feet per second. What is then the velocity of the cwt.?

5. A hammer of mass M lbs. moving at V feet per second, strikes a nail of mass m lbs. and drives it into a piece of wood, the hammer not rebounding from the nail. The resistance of the wood to penetration is N lbs.' wt. Show that the penetration continues for a time MV/Ng of a second.

6. A 4-oz. ball is struck so that it moves with a velocity of 400 feet per second. If the mean force of the blow is 1000 lbs.' wt., for how long does it last?

Summary.

The **momentum** generated by a constant force in a given time is equal to the product of the force and the time.

The momentum generated by a variable force in a given time is equal to the product of the time average of the force and the time.

Impulse. A very large force, like that with which a cricket bat drives a ball, is called an **impulsive force.** It is not necessary or, as a rule, possible to estimate the magnitude of such a force or the time of its action. It is important to know its effect or the momentum which it produces. This is equal to the product of its mean value, or time average, and the time, and is called its **impulse.**

Impulse is a term sometimes used in connexion with ordinary forces, not impulsive, to denote the momentum generated.

Blow means the same as impulse.

If a body A exerts any force on another body B, B exerts an equal and opposite force on A. This is expressed in

Newton's Third Law of Motion.

To every action there is an equal and contrary reaction.

It follows that *if any action takes place between two bodies they communicate to each other equal and opposite momenta.*

Thus, when a gun free to move backwards discharges a shot, the shot and gun have equal momenta in opposite directions.

This principle is true even when the bodies had momenta before the action. In this case the additional momenta communicated are equal and opposite.

It follows that *if any inter-action takes place between two bodies the entire algebraical change in their joint momentum is zero.*

CHAPTER XVII.

KINETIC ENERGY.

Kinetic Energy.—Suppose that a body of mass m, originally at rest, is acted on by a force $\mathbf{f}$ through a distance s, and thus acquires a velocity v.

Let a be the acceleration of the body.

Then $$v^2 = 2as,$$

$$\therefore \quad s = \frac{v^2}{2a};$$

and $$\mathbf{f} = ma.$$

$$\therefore \quad \mathbf{f}s = \tfrac{1}{2}mv^2.$$

Now $\mathbf{f}s$ is the work done on the body in setting it in motion, being the product of the force and the distance through which it acts.

Hence the work done on the body of mass m to give it a velocity v is equal to

$$\tfrac{1}{2}mv^2.$$

Again, suppose the body, moving at first with velocity v, to be brought to rest by the action of a force $\mathbf{f}$, opposite to the direction in which it is moving, in a distance s. The body thus exerts a force $\mathbf{f}$ *in* the direction in which it is moving, and does an amount of work in coming to rest equal to $\mathbf{f}s$.

But if a is the acceleration of the body, measured in the direction *opposite* to the motion, so that, since the velocity is diminishing, a is a positive quantity;

$$v^2 = 2as,$$

and $$\mathbf{f} = ma.$$

$$\therefore \quad \mathbf{f}s = \tfrac{1}{2}mv^2.$$

The work done or given out by the body in coming to rest is, therefore, equal to $\frac{1}{2}mv^2$.

Thus, the quantity $\frac{1}{2}mv^2$ is the measure of the work that must be done on the body to give it the given velocity, or of the work that can be got out of the body as it loses the given velocity and comes to rest. $\frac{1}{2}mv^2$ is, therefore, the work which is stored up in the body of mass m when it has velocity v.

Now the work stored up in a body is also called the energy of the body. The energy of the body in this case is thus $\frac{1}{2}mv^2$.

This energy is possessed by the body because of its state of motion, or in virtue of its motion. Hence it is called **kinetic energy**.

The phrase kinetic energy is often written shortly K.E.

Thus we have the formula

$$\text{K.E. of body} = \tfrac{1}{2}mv^2.$$

This expression for the K.E. gives it, of course, in terms of the same unit as that in terms of which $\mathbf{f}s$ measures the work, that is, in absolute units.

Absolute Units of Work.—It is convenient to have names for the absolute units of work in the two standard systems, the F.P.S. and the C.G.S. systems.

The absolute unit of work in the foot-pound-second system is the work done by a force of a poundal acting through a foot ; it is called the *foot-poundal.*

The absolute unit of work in the centimetre-gram-second system is the work done by a force of a dyne acting through a centimetre ; it is called the *erg.*

Thus, if the quantities are all measured in F.P.S. units $\frac{1}{2}mv^2$ is the K.E. in foot-poundals ; if the quantities are in C.G.S. units $\frac{1}{2}mv^2$ is the K.E. in ergs.

Kinetic energy being by definition work, can, of course, be measured in terms of any unit which is a unit of work.

Thus, suppose we require the number of foot-pounds of energy in a body of mass m lbs. moving with a velocity of v feet per second.

K.E. of body $= \frac{1}{2}mv^2$ foot-poundals.

But one foot-poundal $= \frac{1}{32}$ of one foot-pound.

$\therefore$ K.E. of body $= \dfrac{mv^2}{64}$ foot-pounds.

In the same manner, if a body has mass m grams, and moves with a velocity of v cms. per sec., its K.E. is

$$\tfrac{1}{981} \text{ of } \tfrac{1}{2}mv^2 \text{ gram-centimetres.}$$

In general, if in any case g is the acceleration due to gravity, the expression for the K.E. of a moving body in gravitation units is

$$\frac{mv^2}{2g}.$$

EXAMPLE.—What is the kinetic energy in ergs of a mass of 50 grams moving at 8 metres per second?

$$\begin{aligned} \text{Mass} &= 50 \text{ grams.} \\ \text{Velocity} &= 800 \text{ cms. per sec.} \\ \therefore \text{ K.E.} &= \tfrac{1}{2} \,.\, 50 \,.\, (800)^2 \text{ ergs} \\ &= \mathbf{16000000 \text{ ergs.}} \end{aligned}$$

EXAMPLE.—Find, in foot-tons, the kinetic energy of a train of 90 tons, moving at 20 miles per hour.

$$\begin{aligned} \text{Mass} &= 90 \times 2240 \text{ lbs.} \\ \text{Velocity} &= \tfrac{22}{15} \times 20 \text{ feet per sec.} \\ &= \tfrac{88}{3} \text{ feet per sec.} \\ \therefore \text{ K.E.} &= \tfrac{1}{2} \,.\, 90 \,.\, 2240 \,.\, (\tfrac{88}{3})^2 \text{ foot-poundals} \\ &= \tfrac{1}{64} \,.\, 90 \,.\, 2240 \,.\, (\tfrac{88}{3})^2 \text{ foot-pounds} \\ &= \tfrac{1}{64} \,.\, 90 \,.\, (\tfrac{88}{3})^2 \text{ foot-tons} \\ &= \mathbf{1210 \text{ foot-tons.}} \end{aligned}$$

[It is clear from this example that the K.E. of m tons moving at v feet per second is

$$\frac{mv^2}{2g} \text{ foot-tons.]}$$

EXAMPLE.—A bullet of $\frac{1}{2}$ oz. mass, moving with a velocity of 800 feet per second, strikes a piece of wood and penetrates $1\frac{1}{2}$ inches. What is the resistance to penetration?

Let **f** poundals be the required force.

$$\text{K.E. of bullet} = \tfrac{1}{2} \,.\, \tfrac{1}{32} \,.\, (800)^2 \text{ foot-poundals.}$$

Work done by bullet against resistance in coming to rest

$$=\mathbf{f}\,.\,\tfrac{1}{8} \text{ foot-poundals.}$$

$$\therefore\ \mathbf{f}\,.\,\tfrac{1}{8}=\tfrac{1}{2}\,.\,\tfrac{1}{32}\,.\,(800)^2.$$

$$\mathbf{f}=\frac{(800)^2}{8}=80000.$$

The required resistance is

80000 poundals or **2500 pounds' weight.**

[This question, and many others of the same kind, can be done without the aid of the formula for kinetic energy. This could be done by means of the formulae $v^2=u^2+2as$ and $\mathbf{f}=ma$; and similar questions have already been done in this way. But the formula for kinetic energy enables us to do such questions more quickly.]

EXAMPLE.—A train of mass 150 tons, moving with a velocity of 30 miles an hour is brought to rest in 100 yards. What is the force of resistance in tons' weight?

Let the force be **P** tons' weight.

Mass $=150\,.\,2240$ lbs.

Velocity $=44$ feet per second

$\therefore$ K.E. $=\frac{1}{2}\,.\,150\,.\,2240\,.\,(44)^2$ foot-poundals.

Energy expended $=\mathbf{P}\,.\,2240\,.\,32\,.\,300$ foot-poundals.

$$\therefore\ \mathbf{P}\,.\,2240\,.\,32\,.\,300=\tfrac{1}{2}\,.\,150\,.\,2240\,.\,(44)^2.$$

$$\mathbf{P}=\frac{(44)^2}{4\,.\,32}=\frac{121}{8}=15\tfrac{1}{8}.$$

Force is **$15\frac{1}{8}$ tons' weight.**

EXAMPLE.—A body of mass m kilograms moves with a velocity of v metres per second. Find its kinetic energy in kilogram-metres.

Mass $=1000m$ grams.

Velocity $=100v$ cms. per sec.

$\therefore$ K.E. $=\frac{1}{2}\,.\,1000m\,.\,(100v)^2$ ergs

$$=\frac{1}{2g}\,.\,1000m\,.\,(100v)^2 \text{ gram-centimetres.}$$

But 1 kilogram-metre = 1000 . 100 gram-centimetres.

∴ K.E. of body

$$=\frac{1}{2g} . 1000m . (100v)^2 \div (1000 . 100) \text{ kilogram-metres}$$

$$=\frac{100mv^2}{2g} \textbf{ kilogram-metres.}$$

It might appear at first that, since K.E. of m grams moving at v cms. per sec. is $mv^2/2g$ gram-cms., K.E. of m kilos. moving at v metres per sec. should be $mv^2/2g$ kilogram-metres. But notice that mass is increased 1000-fold, and v^2 is increased 10,000-fold. Thus K.E. is increased 10,000,000-fold, and the unit is only increased 100,000-fold. Therefore, the measure of the K.E. is 100 times as great as in the first case.

If in the second case we took g to denote the acceleration due to gravity in metres per second per second, so that it would be about 9·81, *then* the measure of the K.E. in kilogram-metres would be

$$\frac{1}{2g}mv^2.$$

Exercises XVII. a.

1. Find in foot-poundals and in foot-pounds the kinetic energies of
   (1) 56 pounds moving 10 feet per second.
   (2) $\frac{1}{4}$ ounce moving 2000 feet per second.
   (3) A ton moving 2 feet per second.

2. Find in ergs and in gram-centimetres the kinetic energies of
   (1) 50 grams moving at ·4 metre per second.
   (2) 250 kilograms moving at 40 metres per second.

3. Find in kilogram-metres the energy of a mass of 1500 kilos. moving at 24 metres per second.

4. Compare the kinetic energies of 50 pounds moving at 30 miles an hour and 5 kilos. moving at 6 metres per second.

5. If two bodies are moving with equal momenta, show that their kinetic energies are (1) proportional to their velocities, (2) inversely proportional to their masses.

6. If a bullet discharged with a certain velocity will penetrate 3 inches into a piece of wood, find how far it would penetrate if discharged with half the velocity.

7. A bullet discharged with a certain velocity penetrates 3 inches into a piece of wood. How far will a bullet of half the mass penetrate if it is discharged with the same velocity and if the resistance to penetration is the same as in the case of the other bullet?

8. How many foot-tons of work are done by an engine in getting up a velocity of 20 miles an hour in a train of 90 tons, frictional resistance to motion being neglected?

Increase of Kinetic Energy is equal to Work done by a Force.—Consider now the more general case of a body of mass m, moving in a straight line and having its velocity increased from u to v by the action of a force f upon it, as it moves through a distance s in the sense of the force.

Let a be the acceleration measured in the sense of $\mathbf{f}$, so that a is positive.

Then $$v^2 - u^2 = 2as;$$
and $$\mathbf{f} = ma.$$
$$\therefore \quad \mathbf{f}s = \tfrac{1}{2}m(v^2 - u^2).$$

$\frac{1}{2}mv^2$ and $\frac{1}{2}mu^2$ are the original and final kinetic energies of the body, and $\mathbf{f}s$ is the work done on it. Hence the increase in the kinetic energy of the body is equal to the work done on it by the force.

This is, of course, what we might naturally have expected from what we know about kinetic energy; but it is better to deduce this important result from the fundamental formulae.

It may be inferred from the above result that, when s is negative, or the body moves in the direction opposite to the force, thus meeting with a resistance to its motion and having negative work done on it, the *diminution* of its kinetic energy is equal to the numerical value of this work.

It will be better, however, to give a formal proof of this important proposition.

Suppose a body of mass m to be moving in a straight line and to have its velocity reduced from u to v by the action of a force $\mathbf{f}$ against its motion while it moves through a distance s.

Let a be the acceleration measured in the direction of $\mathbf{f}$, so that a is positive.

Then, since the distance travelled in the direction in which a is measured is $-s$,

$$v^2 - u^2 = -2as,$$

or

$$u^2 - v^2 = 2as;$$

and

$$\mathbf{f} = ma.$$

$$\therefore \ \mathbf{f}s = \tfrac{1}{2}m(u^2 - v^2).$$

Thus, the loss of kinetic energy is equal to the work done by the body against the force.

It is very easy to remember these two results, and they are what we should naturally expect, knowing that kinetic energy means work stored up in a body. They are here rigidly proved, however, by means of the fundamental formulae.

We may state the results in these words :

The increase in the kinetic energy of a body is equal to the work done on it by a force which tends to increase its velocity.

The decrease in the kinetic energy of a body is equal to the work which it does against a force which tends to diminish its velocity.

It must, of course, be borne in mind that the work is to be measured in absolute units, as is obvious, since the formula $\mathbf{f} = ma$, which is used, refers to absolute units. Thus, if we are using the F.P.S. system, the work is in foot-poundals ; if the C.G.S. system, the work is in ergs.

EXAMPLE.—Through what distance must a force of 4 dynes act on a kilogram to change its velocity from 5 to 10 metres per second ?

Let the distance be s cms.

Increase in K.E. of body $= \frac{1}{2} . 1000\{(1000)^2 - (500)^2\}$ ergs.

Work done on body $= 4 . s$ ergs.

$$\therefore \ 4s = \tfrac{1}{2} . 1000\{(1000)^2 - (500)^2\},$$

$$s = \frac{1000 . 1500 . 500}{8} = 93750000.$$

Required distance is **937500 metres.**

EXAMPLE.—A bullet of mass $\frac{3}{4}$ oz. in passing through a 2 inch board has its velocity brought down from 1600 to 1100 feet per second. What is the resistance offered by the board ?

Let **f** poundals be the resistance.

Loss of K.E. $= \frac{1}{2} \cdot \frac{1}{16} \cdot \frac{3}{4}\{(1600)^2 - (1100)^2\}$ foot-poundals.

Energy expended against resistance $= f \cdot \frac{1}{6}$ foot-poundals.

$$\therefore \ \mathbf{f} \cdot \tfrac{1}{6} = \tfrac{1}{2} \cdot \tfrac{1}{16} \cdot \tfrac{3}{4}\{(1600)^2 - (1100)^2\}.$$

$$\mathbf{f} = \frac{6 \cdot 3 \cdot 2700 \cdot 500}{8 \cdot 16} = \frac{759375}{4}.$$

Resistance is **5933 lbs.' wt., about.**

Example.—A body of mass 2 lbs. is moving in a horizontal straight line running west and east under the action of a constant force of 2 poundals to the west. At a certain instant it is moving with a velocity of 4 feet per second eastwards. What will its velocity be when it is 4 feet to the east of this point?

Let v feet per second eastwards be the required velocity.

Loss of energy of body $= \frac{1}{2} \cdot 2(4^2 - v^2)$ foot-poundals.

Energy expended against resistance $= 2 \cdot 4$ foot-poundals.

$$\therefore \ \tfrac{1}{2} \cdot 2(4^2 - v^2) = 2 \cdot 4,$$

$$16 - v^2 = 8,$$

$$v = \pm 2\sqrt{2}.$$

The velocity is **$2\sqrt{2}$ feet per second to the east**, or **$2\sqrt{2}$ feet per second to the west.**

Either of these solutions is correct, and both should be given. The kinetic energy of the body at the given point is known because we know how much energy is spent against the resistance, and kinetic energy depends upon magnitude of velocity, but not upon its direction.

The explanation is that the body travels through the point in question towards the east till it comes to rest for an instant, and then, under the action of the force, which now does positive work on it, travels to the west and passes through the point again with the same velocity as it had in passing through it the first time. The nett amount of energy that has been spent against the resistance when the body is passing through the given point is the same in each case. In the second place more energy has been spent up to the instant that the body comes to rest or reaches its furthest position eastwards, but some has been recovered in returning.

Exercises XVII. b.

1. What resistance must act on a 7 pound mass moving at 20 feet per second to reduce its velocity by one half in 10 feet?

2. If a force of 2 poundals acts on a mass of 4 pounds, moving at first with a velocity of 3 feet per second, through a distance of 7 feet, what is then the velocity of the mass?

3. Through what distance must a kilogram with a velocity of 100 cms. per second move against a resistance of 1 dyne to have its velocity reduced by 1 cm. per second?

4. Through what distance must a kilogram with a velocity of 50 cms. per second move against a resistance of 1 dyne to have its velocity reduced by 1 cm. per second?

Formula applicable to both cases.—The two cases, in which the kinetic energy of a body is increased by having work done on it and diminished through its doing work against a resistance, may be represented by means of a single formula; and some students will find it more convenient to use one formula for the two cases, although they may be treated separately as already shown. The formula is that first given,

$$\mathbf{f}s = \tfrac{1}{2}m\,(v^2 - u^2).$$

This may be used to refer either to the case in which the kinetic energy of a body is increased by a force acting in the sense of the motion or to the case in which the kinetic energy is diminished by a force acting against the motion, by paying due regard to the signs of the quantities.

Always count $\mathbf{f}$ positive and measure s in the sense in which $\mathbf{f}$ acts.

Then if the motion is in the sense of $\mathbf{f}$, s is positive and $\mathbf{f}s$ represents positive work done on the body.

If the motion is opposite to $\mathbf{f}$, s is negative, and $\mathbf{f}s$ represents a negative quantity of work done on the body, or of energy given to the body, and is numerically equivalent to the work done by the body against the resistance.

In the first case $\frac{1}{2}m(v^2 - u^2)$ is the increase of energy and is equal to the work $\mathbf{f}s$.

In the second case $\frac{1}{2}m(v^2 - u^2)$ is a negative quantity and is numerically equivalent to the loss of kinetic energy.

Thus, in each case the equation holds, both sides being positive in the first, and both negative in the second case.

Cases may occur in which a body under the action of a constant force in the straight line of its motion moves at first in the sense opposite to the force and afterwards in the same sense; for the force may at first destroy the body's velocity and then generate an opposite velocity.

A simple and familiar example is when a body is thrown vertically upwards. It is acted upon by a constant downward force, its weight, which first brings it to rest and then gives it a downward velocity.

At first work is done against the force, and afterwards, as the body returns, work is done by the force. The nett amount of work that has been done against the force in any position of the body depends only on the position and not on the sense in which the body is moving. Thus, the loss in kinetic energy depends only on the position.

Because of the importance of the subject the various cases have been dealt with in detail. But the following compact proof may be given to cover every case.

Equation of Energy.—Suppose a body of mass m to be moving in a straight line under the action of a constant force $\mathbf{f}$ in the same line.

Let u be the initial velocity of the body and v the final velocity when it has described a distance s from its original position, all these quantities being measured in the sense in which $\mathbf{f}$ acts, so that any of them may be positive or negative.

Let a be the acceleration of the body measured in the sense of $\mathbf{f}$, so that a is positive.

Then $$\mathbf{f}=ma,$$

and $$v^2=u^2+2as.$$

From these we get $$\mathbf{f}s=\tfrac{1}{2}m(v^2-u^2).$$

This equation, coming from the above two which are always true, is itself always true.

If s is positive, work is on the whole done by $\mathbf{f}$, and $\mathbf{f}s$ is positive, and the K.E. is increased.

If s is negative, work is on the whole done against $\mathbf{f}$, and $\mathbf{f}s$ is negative, and the K.E. is diminished.

The equation $$\mathbf{f}s=\tfrac{1}{2}m(v^2-u^2)$$

may be called **the equation of energy.**

It states that :

The amount of work done on the body is equal to the increase of kinetic energy (due regard being paid to the signs of both these quantities).

Even if the equation is not used in this form in cases of loss of energy, it is convenient to have one standard form to refer to as the equation of energy.

We shall now exemplify the two ways of treating a question on loss of kinetic energy in the following problem.

EXAMPLE.—Find, by the principles of kinetic energy the velocity of a body which is thrown upwards with a velocity of 45 feet per second when it has reached a height of 10 feet above its starting point.

Let mass of body $= m$ lbs.
Downward force of 32 m poundals acts on body.
Let v feet per second be required velocity.

First Method.

Loss of kinetic energy $= \frac{1}{2}m(\overline{45}^2 - v^2)$ foot-poundals.

Work done against weight $= 32m \,.\, 10$ foot-poundals.

$$\therefore\ \tfrac{1}{2}m(\overline{45}^2 - v^2) = 32m \,.\, 10.$$
$$v^2 = \overline{45}^2 - 640 = 1385.$$
$$v = \pm 37{\cdot}2.$$

Second Method.

Increase of kinetic energy $= \frac{1}{2}m(v^2 - \overline{45}^2)$ foot-poundals.

Work done on body by weight $= -32m \,.\, 10$ foot-poundals.

$$\therefore\ \tfrac{1}{2}m(v^2 - \overline{45}^2) = -32m \,.\, 10.$$
$$v^2 = \overline{45}^2 - 640 = 1385.$$
$$v = \pm 37{\cdot}2.$$

The velocity of the body is **37·2 feet per second.** The two signs indicate that the body may be going up or coming down.

The increase of kinetic energy mentioned in the second method is, of course, a negative quantity, as is obvious to begin with.

Example.—A bullet weighing half an ounce is projected vertically upwards with a velocity of 1600 feet per second and rises 3000 feet. What work, in foot-pounds, is done against the resistance of the air in the ascent?

Initial kinetic energy of bullet

$$=\tfrac{1}{2}\cdot\tfrac{1}{32}(1600)^2 \text{ foot-poundals}$$
$$=\tfrac{1}{32}\cdot\tfrac{1}{2}\cdot\tfrac{1}{32}\cdot(1600)^2 \text{ foot-pounds}$$
$$=\tfrac{1}{2}(50)^2 \text{ foot-pounds}$$
$$=1250 \text{ foot-pounds.}$$

This is expended against gravity and air resistance.

$$\text{Work done against gravity} = \tfrac{1}{32}\cdot 3000 \text{ foot-pounds}$$
$$= 93\tfrac{3}{4} \text{ foot-pounds.}$$

$\therefore$ work done against resistance of air $= \mathbf{1156\tfrac{1}{4}}$ **foot-pounds.**

This is an example of the action of more than one force on a body causing a change in its kinetic energy.

Any questions of this sort are readily solved by noticing that work done by a force helping its motion is a contribution to its kinetic energy, and work done by the body against a resistance is taken from its kinetic energy.

Exercises XVII. c.

1. Find the resistance to motion per ton of a train which brings it to rest in 220 yards when it is moving at 60 miles an hour.

2. If a train has its velocity reduced from 60 to 20 miles per hour by the action of a constant resistance in 200 yards, in what further distance will it be brought to rest?

3. A falling stone passes a point with a velocity of 30 feet per second. With what velocity did it pass a point 20 feet higher up? Interpret the double sign.

4. A body is thrown vertically upwards with a velocity of 40 metres per second. Find by the principle of energy how high it will rise, neglecting the resistance of the air.

5. A body of mass 70 grams in falling through 10 metres acquires a velocity of 13·5 metres per second. Find the number of kilogram-metres of work done against the resistance of the air.

Summary.

The work that must have been done on a body of mass m to give it velocity v, or the work which a body of mass m having velocity v will do against opposing resistance in coming to rest, is called the **kinetic energy** of the body.

K.E. of body is

$$\frac{1}{2}mv^2 \text{ in absolute units ;}$$

$$\frac{1}{2g}mv^2 \text{ in gravitation units.}$$

The absolute unit of work or energy in the F.P.S. system is called the **foot-poundal**.

The absolute unit of work or energy in the C.G.S. system is called the **erg**.

If a body is moving in a straight line, and is acted on by a force in that line, then the entire amount of work (algebraical) done on the body by the force is equal to the increase (algebraical) of the body's kinetic energy. This is true whether the work done on the body is positive or negative.

CHAPTER XVIII.

POTENTIAL ENERGY. CONSERVATION OF ENERGY. PERPETUAL MOTION. ENERGY AFTER COLLISION.

The Dynamical Equations.—We have found three equations to use in solving questions relating to the motion of bodies in straight lines under the action of force.

We will now collect these together.

(1) $\mathbf{f}=ma$: **the force equation.**

(2) $\mathbf{f}t=m(v-u)$: **the momentum equation.**

(3) $\mathbf{f}s=\frac{1}{2}m(v^2-u^2)$: **the energy equation.**

Of these the first is the fundamental equation. It has been deduced from Newton's Second Law of Motion and the definition of the absolute unit of force. The other two are deduced from this and the kinematical equations. As we have seen, questions involving the use of (2) and (3) can also be solved by means of (1) and the kinematical equations. But (2) and (3) often afford very convenient and ready solutions in such cases.

Potential Energy.—Imagine a body which has been raised to a height and then let fall. On reaching the ground it possesses kinetic energy, which it had not at first. But before beginning to fall it possessed the equivalent of this energy in virtue of its position, for it is because of its original position that it is able to acquire its kinetic energy. The body when raised up, and before beginning to fall, is said to possess *potential energy.* The measure of the potential energy given to a body, by raising it through a certain height, is the same as that of the kinetic energy which it will acquire in falling through the same height, no resistances being supposed to act on it.

The potential energy of a body is the energy which it possesses in virtue of its position.

The kinetic energy of a body is the energy which it possesses in virtue of its motion.

Suppose a mass m pounds is raised up to a height h feet. The kinetic energy that it would acquire in falling is equal to the work that would be done in the fall by its weight, and this is equal to the work done against its weight in raising it. Or, we may say that the potential energy given to it is equal to the work done on it in raising it against the action of its weight. Therefore,

$$\text{POTENTIAL ENERGY} = \begin{cases} mh \text{ foot-pounds;} \\ \text{or, } mgh \text{ foot-poundals.} \end{cases}$$

In general, if a mass m is raised to a height h,

$$\text{POTENTIAL ENERGY} = \begin{cases} mh \text{ gravitation units;} \\ \text{or, } mgh \text{ absolute units.} \end{cases}$$

Examples of Potential Energy.—Many other examples of potential energy may be given besides those which occur in Mechanics. The following are cases of bodies possessing potential energy: a bent spring; the wound-up spring of a watch, which, by uncoiling, is capable of keeping the watch in motion for some time; compressed air, which by expanding can do work; gunpowder, which by exploding can set a bullet in motion. Combustible bodies possess potential energy of a sort, namely, chemical energy; for instance, the carbon of coal and the oxygen of the air, with which the carbon combines in burning, possess this potential energy between them, and in burning it is developed. It may be changed in part into useful mechanical work by means of a steam engine. Or, if the combustion merely generates heat, this is a form of kinetic energy, for there is little doubt that heat consists of a violent state of agitation of the molecules of the bodies in which it resides.

Conservation of Energy.—When a body falls it loses potential energy, but the amount of potential energy lost at each instant, supposing that no resistance acts on it, is exactly equal to the amount of kinetic energy acquired. Or, the entire amount of mechanical energy remains constant.

When the body strikes the ground and comes to rest it loses all its energy of both sorts. But the energy has not been destroyed. The impact develops heat, and heat, as we have seen, is a form of energy. Also in falling through the air, in consequence of the resistance of the air, the body is constantly losing mechanical energy (meaning by this the sum of its potential and kinetic energies); but at the same time heat is being developed by the friction of the body against the air.

The heat developed at first resides partly in the body and partly in the air or the ground; and is ultimately dissipated into surrounding space. And this heat is a quantity of energy exactly equal to the mechanical energy lost by the body. This energy has been lost, perhaps, as far as getting useful work from it is concerned; but it has not been destroyed; it still exists in the universe. It may be lost commercially, but it cannot be lost physically. Neither can the energy given to the body be made; it must be drawn from some source of energy already existing. When the body is raised up energy is given to it which it had not before, but this energy must have previously existed somewhere, although it may not be missed from anywhere. If a person raises a ball by a muscular effort he uses up some of the chemical energy of his body which has been derived from his food. If a body is raised by means of a steam-engine the energy used is derived from the chemical energy of the coal.

There are in nature many forms of energy of which we have here mentioned what may be considered the principal ones, mechanical energy, heat, chemical energy. Other forms are electrical energy, magnetic energy, molecular energy (as in the case of a bent spring).

A given quantity of energy can be transformed from one of these forms into another, just as we have seen that energy of motion can produce heat, and heat can produce energy of motion, as in the case of a locomotive; but the energy can never be increased or diminished in amount. Some forms are more useful than others from a practical point of view; but from a physical point of view they must all be considered as energy. Every agent which does work, such as a living being or a steam-engine, is simply acting as a transformer of energy and changing it from one form to another.

Experience shows that however energy may be changed from one form to another it can neither be created nor destroyed, and a given quantity of it cannot be increased or diminished in amount.

The amount of energy in any region may be changed, for energy can pass out of and into it. The energy of the earth changes in amount, for heat is radiated to and away from the earth.

The proposition here enunciated is sometimes expressed in the following way.

This is known as **The Doctrine of the Conservation of Energy** :

The amount of energy in the universe is unalterable.

An application has been made of this principle in considering the action of machines. The work applied to a machine, or the energy expended on it, is equal to the work that is got from it together with the energy spent in overcoming friction, which last is transformed into heat. In the case of a theoretically perfect machine there is no friction, and the work done on the machine is equal to the work got from it.

Perpetual Motion.—Many attempts have been made from time to time to invent a machine which will keep itself in motion, and not only that, but set other things in motion, and so do useful work without the expenditure of any energy on it. But it is clear from the principle of the conservation of energy that all such attempts must be futile, as no energy can be got from an agent which is not supplied to it in some form. In fact, no machine could be made which would even keep itself in motion without doing work ; for there must always be some amount of friction of the parts which would bring it to rest ; so that if the machine were set going its kinetic energy would be transformed into heat.

Motion on Inclined Plane.—Suppose a body of mass m to slide down a length l of an inclined plane without friction, the corresponding vertical height being h. Consider the change in the kinetic energy of the body.

The resolved part of the body's weight down the plane is mgh/l.

$\therefore$ the work done by the force in the distance l is mgh (in absolute units).

∴ increase in kinetic energy $= mgh$.

We may notice that the body descends by a vertical height h, or falls to a point which is at a level lower than its first position by the distance h; and the increase in kinetic energy is the same as if the body had fallen through this distance in a vertical straight line.

Thus, if a body descends through a given distance, whether it is by falling freely or by sliding down a smooth plane, the increase in its kinetic energy is the same; and so we must consider that the decrease in the potential energy is the same in both cases.

In a similar manner, if a body is sliding up an inclined plane without friction, in passing up through a vertical height h it loses a quantity of kinetic energy, mgh absolute units. When all its kinetic energy is lost it comes to rest for an instant and then slides down.

Energy of Pendulum.—In the case of a pendulum, or heavy body suspended by a string and oscillating to and fro, the energy is constantly changing from potential to kinetic and back again. Neglecting the resistances which ultimately bring the body to rest, the entire energy remains unchanged in amount. When the body is at the extreme end of its swing, and is therefore in its highest position, all its energy of the pendulum is potential. As it falls it loses potential and acquires kinetic energy. In its lowest position, when the string is vertical, it has lost all its potential energy and its kinetic energy is greatest, so that its velocity is then greatest. As it rises from this position it loses kinetic and regains potential energy; and so on.

Bodies connected by String over Pulley.—Suppose two bodies, of masses m, m', to be connected by a string passing over a smooth light pulley. Let m be greater than m'.

The bodies being free to move the whole of the work done by gravity in any motion is spent in generating kinetic energy in them.

Suppose that m descends and m' ascends a distance h.

A force mg acts in the sense of the motion, and a force $m'g$ opposite to the motion.

Therefore the work done is $mgh - m'gh$.

[The work done is the same as if the bodies were moving in one straight line with a force mg acting to help the motion and a force $m'g$ in the opposite sense.]

Let v be the velocity acquired by the bodies from rest in this motion.

Then the kinetic energy acquired is $\frac{1}{2}(m+m')v^2$.

$$\therefore \ \frac{1}{2}(m+m')v^2=(m-m')gh.$$

The case of two bodies, one of which moves horizontally, and is connected by a string to the other, which moves vertically downwards, may be considered in a similar manner, work in this case being done by the weight of one body only, the falling one.

Exercises XVIII. a.

1. Bodies slide down various inclined planes of the same height but of different lengths. Show that they all reach the feet of the planes with the same velocity.

2. A body hung at the end of a string 10 feet long is drawn aside 6 feet, measured in a horizontal direction, from the original vertical position of the string, and is then let go. With what velocity does it pass through its original position?

3. A body of mass 20 pounds slides 100 feet down an inclined plane, of slope 1 in 50, and acquires a velocity of 6 feet per second. It is retarded by the action of a constant frictional resistance. What work is done against this resistance, and what is the value of the resistance?

4. A body of mass 10 pounds slides down an inclined plane 100 yards long with a slope of 1 in 25. The frictional resistance of the plane is $\frac{1}{300}$ of the weight of the body. If the body has a velocity of 26 feet per second at the foot of the plane, what is the work done against the resistance of the air?

5. An engine pulls a train of 40 tons up an incline of 1 in 80 through 400 feet, while the velocity of the train falls off from 20 to 15 miles an hour. If the resistances to motion be 10 lbs.' wt. per ton, what is the entire amount of work done by the engine?

6. Two bodies of masses 24 and 25 ozs. are connected by a string which passes over a smooth light pulley. If they are left free to move, find their velocity when each has moved through a foot.

7. Two bodies of masses 5 and 7 pounds are connected by a string passing over a pulley. When they have moved through 6 feet they have a velocity of 6 feet per second. How much work in foot-pounds has been expended against friction and in setting the pulley in motion?

8. Two masses, 13 and 11 units, are connected by a string passing over a smooth light pulley; when each has moved through 5 feet they are estimated to have a velocity of 5 feet per second. What value for the acceleration due to gravity does this give?

9. If a gram weight hanging by a fine thread passing over a perfectly smooth light pulley at the edge of a perfectly smooth horizontal table draw a 2 kilogram weight along the table, what is the common velocity when the gram weight has descended 1 metre?

10. A body of mass m is attached to the end of a string. The string passes over a fixed pulley under a movable pulley and has its other end fixed. All the straight parts of the string are vertical. A mass M is attached to the movable pulley. m descends and draws up M. Show that when m has descended by h and has velocity v,

$$v^2(4m+M)=4gh(2m-M).$$

Energy after Collision.—If two bodies, one or both of which are in motion, collide, some of their energy of motion is always lost in the impact. Heat is developed by the impact, the energy of which is exactly equal to that which is lost from the kinetic energy of the bodies.

Suppose, for example, that two equal bodies are moving towards each other with equal velocities in the same straight line. On colliding they will rebound with velocities which are equal to each other, but less than those which the bodies had at first, no matter of what materials the bodies are made. Hence some kinetic energy is lost.

If the two bodies are of such materials that they do not rebound at all, but remain at rest together after the collision, then all their kinetic energy is lost, and has been converted into heat.

Consider the change in kinetic energy when one body impinges on another, and the two move on together.

Suppose a body of mass m, moving with velocity v, to impinge on a body of mass m', which is at rest, so that the two go on moving together. To find the kinetic energy of the system after the impact,

Let v be the joint velocity after impact.

By conservation of momentum,

$$(m+m')\,v=mu.$$

$$v=\frac{mu}{m+m'}.$$

New kinetic energy

$$=\tfrac{1}{2}(m+m')\,\frac{m^2u^2}{(m+m')^2}$$

$$=\frac{m^2u^2}{2(m+m')}.$$

Also, ratio of new to old kinetic energy

$$=\frac{m^2u^2}{2(m+m')}\div\frac{mu^2}{2}$$

$$=\frac{m}{m+m'}.$$

These results are not intended to be remembered, but merely as illustrations of the subject.

The student must carefully avoid the error of supposing that after collision bodies have the same kinetic energy as before. Kinetic energy is always diminished by the collision. The entire energy is unchanged, but a part ceases to be mechanical energy. And, of course, the entire momentum estimated in any given direction remains unchanged.

When one body strikes another and the two move on together, overcoming a frictional resistance, as when a hammer strikes a nail and drives it into a piece of wood, kinetic energy becomes converted into heat in two distinct ways.

First, some kinetic energy is transformed into heat by the impact; the two bodies begin to move on together with momentum equal to the momentum which the moving one had just before the impact, but with a smaller kinetic energy.

Secondly, the kinetic energy, or a part of it, which the two bodies have when they begin to move together is transformed into heat in overcoming the resistance.

The Pile-driver.—The action of the pile-driver illustrates well the joint energy of bodies after collision. The driver is a heavy weight which falls on the top of the pile. A portion of the energy of the driver is at once converted into heat by the collision, and the driver and pile begin to move on with the remainder of the energy, which is used up in overcoming the resistance of the earth to penetration.

The illustration shows the form of pile-driver called the **monkey-engine.** The weight Q which falls on the pile is called the monkey.

Q is hauled up by means of a winch with two handles A, A. When Q is at the top the toothed wheels are thrown out of gear with each other by means of the lever CDE, so that Q descends, uncoiling the rope from the barrel G of the winch.

Another plan is to have an arrangement at the top of Q which causes it to be released when drawn up to top. This has the advantage of allowing Q to acquire a greater velocity in its fall.

FIG. 146.—Pile driver (monkey engine).

EXPERIMENT 42.—Take two blocks of wood, each about 3 ins. × 3 ins. × 12 ins. Suspend each by two strings, one near each end, the strings being of equal length and about 4 feet long, so that when the block hangs at rest it is horizontal and the strings are vertical. Also let the blocks be so hung that when they are at rest they are in line with each other, the two nearest ends being about $\frac{1}{8}$ in. apart. Now each block can be made to swing in the vertical plane of its strings, always remaining horizontal as it swings.

Now, if one of the blocks is drawn aside so that it rises by a height h ft. from its lowest position, and then released, it will reach that position with a velocity $\sqrt{64h}$ feet per second. And, in the same way, if it is observed to swing up to a height h, it must have passed through its position of rest with a velocity $\sqrt{64h}$ feet per second.

In experiments h will be a small quantity, and the best way to determine it accurately in any case is to observe the *horizontal* distance by which the block is displaced from its position of rest. If this is k and the length of each string is l, then in the displaced position the lowest ends of the strings are at a vertical depth below the highest ends equal to $\sqrt{l^2 - k^2}$; $\therefore\ h = l - \sqrt{l^2 - k^2}$.

Thus, if the strings are 4 feet long and the block is drawn aside 5 inches, it is easily calculated that it is raised by ·022 ft., and acquires on reaching its lowest position a velocity of about 1·2 feet per second.

To measure the horizontal displacements, a scale may be placed horizontally beside the blocks. With the help of this it can easily be observed how far the block is drawn aside or how far it moves in its swing before beginning to return.

In order to make the blocks move together after collision, a couple of small spikes may be placed in the end of one, with the sharp points protruding so that these may penetrate the other block. Or, one of the blocks may be provided with a small spring catch at its top front edge, which will slide over a pin near the edge at the top of the other, and thus fasten the blocks together.

The masses of the blocks may be varied by placing weights upon them, strips being nailed along the upper faces of the blocks to prevent the weights from moving at the instant of the collision.

Care must be taken that the weights do not overbalance. This may be done by making each string to terminate in a loop which goes round the block, the knot of the string being a few inches above the block.

Let the masses of the blocks be m, m'. Let m be drawn aside and strike m'. By observing the distance to which m was drawn and the distance to which they move together on the other side, we can determine the velocities, v, v' of m just before collision, and of the two together just after. The corresponding momenta are mv and $(m+m')v'$, which should be equal.

Experiment 43.—Arrange the blocks without the spikes or catch so that they separate after collision. It will now be necessary for two observers to determine the distances to which the blocks swing.

For large values of m, m will follow m'; for small values of m, m will have its direction of motion reversed by the collision.

Let v', v'' be the calculated velocities of m and m' after collision.

The joint momentum of m and m' after collision is $mv' + m'v''$ or $mv' - m'v''$, according as m follows m' or rebounds.

In each case this should be equal to mv.

EXAMPLE.—A hammer weighing 2 lbs., moving with a velocity of 6 feet per second, strikes a $\frac{1}{4}$ oz. nail and drives it $\frac{1}{4}$ in. into a piece of wood. What is the resistance, supposed constant, of the wood to penetration, and how long does the penetration last? The hammer does not rebound from the head of the nail.

Let the velocity with which hammer and nail begin to move after impact be v feet per second.

Then, by conservation of momentum,

$$\left(2+\frac{1}{4\,.\,16}\right)v=2\,.\,6,$$

$$v=\frac{128\,.\,6}{129}.$$

Let the required resistance be **F** poundals.

Then, since kinetic energy = work done against resistance,

$$\frac{1}{2}\left(2+\frac{1}{4\,.\,16}\right)\left(\frac{128\,.\,6}{129}\right)^2=\mathbf{F}\,.\,\frac{1}{4\,.\,12}.$$

$$\therefore\ \mathbf{F}=48\,.\,\frac{129}{128}\,.\left(\frac{128\,.\,6}{129}\right)^2=\frac{48\,.\,128\,.\,36}{129}.$$

$$\text{Required force}=\frac{6\,.\,128\,.\,9}{129}\text{ lbs.' wt.}$$

$$=\mathbf{53{\cdot}58\ lbs'.\ wt.}$$

Let t secs. be the duration of the penetration.

Then, since momentum $2\,.\,6$ units is destroyed by the force **F** poundals in t seconds,

$$\mathbf{F}t=2\,.\,6,$$

i.e.

$$\frac{48\,.\,128\,.\,36}{129}t=12,$$

$$t=\frac{129}{4\,.\,128\,.\,36}$$

$$=\frac{43}{6144}.$$

$$\text{Required time}=\mathbf{\frac{43}{6144}\ secs.}$$

The time could also be found by considering that it is the time with which a body will travel $\frac{1}{48}$ ft. with mean velocity $\frac{1}{2}$

of $\frac{128 \cdot 6}{129}$ feet per second; and this may be done as a check on the result already obtained.

The time thus got is

$$\frac{1}{48} \div \frac{1}{2} \text{ of } \frac{128 \cdot 6}{129} \text{ sec.} = \frac{129}{24 \cdot 128 \cdot 6} \text{ secs.}$$

This agrees with the result already got.

Again, this could have been got first in this way and the force then obtained from the momentum equation.

In this question no account is taken of the weight of the hammer and nail, because there is nothing to indicate how they would act. They would have very little effect in any case; and, if the hammer were driving the nail horizontally, they would have no effect at all.

Example.—A pile-driver of mass 10 tons falls from a height of 9 feet on the head of a pile of mass $\frac{1}{2}$ ton and drives it 6 inches into the ground. What is the resistance of the ground to penetration?

Velocity of driver just before striking pile

$$= \sqrt{2 \cdot 32 \cdot 9} \text{ ft. per sec.} = 24 \text{ ft. per sec.}$$

Driver and pile begin to move together with velocity

$$\frac{10}{10\frac{1}{2}} \cdot 24 \text{ ft. per sec.} = \tfrac{160}{7} \text{ ft. per sec.}$$

Driver and pile have, to start with, kinetic energy

$$\tfrac{1}{64} \cdot 10\tfrac{1}{2} \cdot (\tfrac{160}{7})^2 \text{ foot-tons.}$$

Let the resistance be **P** tons' weight.

Driver and pile move against resultant upward force $\mathbf{P} - 10\frac{1}{2}$ tons' weight through distance $\frac{1}{2}$ foot.

Work done against this resistance is

$$(\mathbf{P} - 10\tfrac{1}{2}) \cdot \tfrac{1}{2} \text{ foot-tons.}$$

$$\therefore\ (\mathbf{P} - 10\tfrac{1}{2})\tfrac{1}{2} = \tfrac{1}{64} \cdot 10\tfrac{1}{2} \cdot (\tfrac{160}{7})^2,$$

$$\mathbf{P} = 10\tfrac{1}{2} + \tfrac{21}{64} \cdot (\tfrac{160}{7})^2$$

$$= 10\tfrac{1}{2} + 171\tfrac{3}{7}$$

$$= 181\tfrac{13}{14}.$$

Required resistance is **$181\frac{13}{14}$ tons' weight.**

Exercises XVIII. b.

1. A 7 lb. mass, moving with a velocity of 10 feet per second, strikes against a 4 lb. mass at rest, and the two move on together. What is their kinetic energy in foot-poundals?

2. A body of mass 60 kilograms, moving with velocity 40 metres per second, strikes against an equal mass at rest, and the two move on together. How many ergs of energy are converted into heat?

3. A half-ounce bullet is fired horizontally into a piece of wood of 20 lbs. mass suspended by strings so that it can swing freely. The wood rises through a vertical height of 7 inches. What was the velocity of the bullet?

4. A 1-oz. ball thrown horizontally against the bob of a pendulum of mass 2 oz. with a velocity of 10 feet per second rebounds with a velocity of 3 feet per second. With what velocity does the pendulum begin to move? and how high does it rise?

5. A truck of mass 4 tons running on smooth rails with a velocity of 8 feet per second strikes another of mass 5 tons at the foot of a slope of 1 in 81. How far up the slope do they run together?

6. A ball slides down the sloping part of a smooth groove, falling vertically 1 foot, and strikes a row of four exactly similar balls on the horizontal part of the groove, which begin to slide off with a velocity of 3 feet per second. Show that the first ball rebounds and ascends the sloping groove to a vertical height of 3 inches.

7. Two masses of 4 and 8 lbs. are connected by a string which passes over a light smooth pulley. The 4 lb. mass drops through 4 feet, and then the string becomes tight, and the other body which was at rest begins to ascend. Find the initial common velocity, and how far the 8 lb. mass will ascend.

8. A body of mass m falls from a height h and then penetrates a depth a into the ground. Show that the resistance to penetration is $m\left(\frac{h}{a}+1\right)$ gravitation units.

9. A body of mass m falls from a height h on the top of a spike of mass m', which it drives a depth a into the ground. Show that the resistance to penetration is $\frac{m^2h}{(m+m')a}+m+m'$ gravitation units.

10. How is the formula $v^2=2fs$ obtained?
Two masses, of 3 lbs. and 5 lbs. respectively, are resting close together on a smooth horizontal plane, and are connected by a string 16 feet long. If the 5 lb. mass is acted upon by a constant horizontal force of 10 poundals, what will be its velocity when the string becomes tight, and with what velocity will the other mass begin to move? (Coll. Precep., 1898.)

11. If a mass of 121 lbs., originally at rest at the bottom of a pit, is raised with a uniform acceleration of 2 feet per second per

second, what will be the kinetic energy of the mass at the end of 5 seconds? and how much work will then have been done in raising it? (Coll. Precep., 1898.)

12. A particle slides in a straight line on a smooth plane against a constant resistance; at a certain point its velocity was 15 feet a second, and at a point 12 feet further on its velocity was 8 feet a second; its mass is 7 lbs.; find its kinetic energy in each position, and the resistance against which it was moving.

State in what units of energy and force your results are expressed, and define those units. (Science and Art, 1897.)

13. A particle, whose mass is 10 lbs., moves in a straight line, and its velocity is changed from 210 feet a second to 90 feet a second; find the numerical value of the change of its kinetic energy. State what units are used in your answer.

If the change is produced by a force equal to 3 lbs. weight ($g=32$), through what distance does the particle move while its kinetic energy is undergoing the change? (Science and Art, 1898.)

14. State the principle of the Conservation of Energy, and use it to find with what velocity a bicycle must start the ascent of a hill 20 feet high in order that it may just reach the top although the rider does no work. (Camb. Sr. Loc., Stat. Dyn. and Hydro., 1896.)

15. A marble weighing one ounce falls from rest from a height of 100 feet and reaches the ground with a velocity of 64 feet per second. Find how much work has been spent against the resistance of the air, expressing the result in foot-pounds. (Camb. Sr. Loc., Stat. Dyn. and Hydro., 1898.)

16. When is a force said to do work, and how is the work measured?

Masses of 10 and 15 ounces are suspended from the ends of the string of an Attwood's Machine; calculate in foot-pounds the total work done on both masses by gravity when the heavier mass descends 3 feet. (Oxford Locals, 1898.)

17. A body, the mass of which is 10 pounds, is moving with a velocity of 50 feet per second. How much is its kinetic energy? How much is its momentum? State the units in which your answers are expressed. If half its energy were given to a body of 20 lbs.' weight, with what velocity would this latter be then moving? (London Matric., 1899.)

18. If a pendulum bob weigh 2 pounds, and in swinging rises to a height of $\frac{3}{10}$ths of an inch at the end of its swing, calculate the potential energy it has received. Assuming that in the descent of the bob during the next half of the return swing all this potential energy is transformed into kinetic energy, calculate the speed it will acquire. How would you propose to measure the height through which the centre of gravity of the bob rises and falls while swinging? [*N.B.*—The acceleration of gravity may be taken as 32 feet per second per second.] (London Matric., 1899.)

Summary.

Potential Energy is the energy possessed by a body **in virtue of its position** ; as in the case of a body raised to a height. This energy can be converted into energy of motion by letting the body fall.

Kinetic Energy is the energy possessed by a body **in virtue of its motion.**

Potential energy of mass m raised to a height h is mh gravitation or mgh absolute units.

There are in nature many other examples of energy stored up in bodies, which may be called potential energy, besides the mechanical potential energy of a raised weight. For instance, the *molecular* energy of a bent spring, the *chemical* energy of coal.

Conservation of Energy. Energy cannot be created or destroyed by any means known to us. It can only pass from one body to another and be transformed from one form into another. When an agent does work it only transforms energy. The entire amount of energy in the universe is unalterable.

Perpetual Motion. Attempts made to produce a machine which shall do work, or even continue to keep itself in motion, without the supply of energy to it from without must ever be futile, because this is contrary to the principle of the conservation of energy.

Motion on Inclined Plane. In the case of a body moving freely on an inclined plane the increase in its potential energy is measured by the product of its weight and the *vertical* height through which it rises, and this is equal to the decrease in its kinetic energy.

Pendulum. In the case of a pendulum swinging to and fro the energy is continually changing from potential to kinetic and back again. But the sum of the energies, supposing that the swings do not die away, remains constant.

Bodies connected by a string passing over a smooth light pulley can have their velocity in any position determined from the fact that the kinetic energy at any instant is equal to the nett amount of work which has been done on them by gravity, or their loss of potential energy.

Energy after Collision. When two bodies collide some of the energy which they possessed is always converted into heat by the impact. The amount that is thus lost will depend on the circumstances of the motion, as well as on the materials of which the bodies are made.

CHAPTER XIX.

RELATIVE VELOCITY AND ACCELERATION. COMPOSITION OF VELOCITIES AND ACCELERATIONS. UNIFORM CIRCULAR MOTION.

Relative Velocity and Acceleration.—If two bodies, A and B, are moving along the same straight line, then the rate at which the distance of A from B is increasing is called the velocity of A relative to B; and the rate at which this relative velocity is increasing is called the acceleration of A relative to B.

Relative velocity and acceleration along a straight line are, of course, susceptible of algebraical signs, just like ordinary velocity and acceleration.

If, for example, distances, etc., are measured to the right, and A is to the right of B, but getting nearer to B, A has a negative velocity relative to B. And if A's velocity of relative approach to B is getting smaller and smaller, so that, if the accelerations remain the same, A will ultimately recede from B rightwards, A has a rightward or positive acceleration relative to B.

EXAMPLE.—Two bodies start from the same point in a straight line at the same instant and move in opposite directions, one with a constant velocity of 7 metres per second, and the other with an initial velocity of 10 metres per second, and an acceleration, in the same direction, of 3 metres per second per second. Find, in metres and seconds, their distance apart, their velocity of separation, and their relative acceleration at the end of 3 seconds.

In three seconds the distances travelled are

$$7 \,.\, 3 = 21 \text{ metres,}$$

and $$10 \,.\, 3 + \tfrac{1}{2} \,.\, 3 \,.\, 3^2 = 43\tfrac{1}{2} \text{ metres.}$$

$\therefore$ distance apart $= $ **64½ metres.**

The velocities are 7 metres per sec., and $10 + 3 \,.\, 3 = 19$ metres per sec.

And these are in opposite directions,

$\therefore$ velocity of separation $=$ **26 metres per sec.**

The first velocity is constant, the second increases at the rate of 3 metres per sec. per sec.

$\therefore$ the velocity of separation increases at the rate of, or the relative acceleration is, **3 metres per sec. per sec.**

Suppose that A is to the right of B, and that A is moving with a velocity of v feet per sec., and B with a velocity of v' feet per sec.

(i) If the velocities are uniform, in one second A and B will move through v and v' feet to the right.

$\therefore$ A will be $v - v'$ feet further from B.

$\therefore$ the rate at which the distance from B to A is increasing is $v - v'$ feet per sec., *i.e.* A's velocity relative to B is $v - v'$ feet per second rightwards.

If A is to the left of B, A is approaching B at the rate $v - v'$ feet per second.

A's velocity relative to B is still $v - v'$ feet per second rightwards.

If the signs of any of the quantities v, v', $v - v'$ are negative, the velocity of A relative to B rightwards is always $v - v'$ feet per second.

(ii) If the velocities are not uniform; then if the velocities were to remain uniform for a whole second and equal to what they are at the instant in question, A would travel through $v - v'$ feet rightward relative to B.

.. at the instant in question A's velocity relative to B is $v - v'$ feet per second rightward.

If the accelerations of A and B to the right are a and a' at any instant, then the rate at which A is acquiring velocity relative to B to the right is $a - a'$.

That is, A's acceleration relative to B to the right is $a - a'$.

As an example of a question solved by the method of relative velocities consider the following:

EXAMPLE.—A body is let fall, and one second after another is thrown down with a velocity of 48 feet per second. In what time after starting will it overtake the first?

In t seconds after the second body starts the velocities of the two bodies are

$$32(t+1) \text{ and } 48+32t \text{ feet per second.}$$

∴ second body approaches the first at the rate of 16 feet per second.

And when the second body starts the first has descended 16 feet.

∴ the second body overtakes the first in **1 second after starting.**

[This question could, of course, also be solved by writing down the distances travelled by the bodies at the end of t seconds after the second starts and equating the expressions.]

Composition of Velocities.—A body may have two separate independent velocities at the same time, as for instance when the body is carried in a carriage and is at the same time moved relatively to the carriage. The body will have a resultant velocity made up of, or compounded of, the velocity of the carriage and the velocity of the body relatively to the carriage. To find its resultant velocity from the other velocities is called *compounding* these velocities.

If the motion of the body relatively to the carriage is in the same direction as the motion of the carriage, that is, if it is parallel to the line along which the carriage is moving, then, if both the motions are in the same sense, it is clear that the rate at which the body is moving away from or towards a fixed point in the line of motion is the sum of the velocities; if they are in opposite senses the rate is the difference of the velocities.

Thus velocities along the same straight line are compounded by adding them if they are in the same sense and by subtracting them if they are in opposite senses.

Triangle of Velocities.—The more general case in which the two component velocities are not in the same direction may be illustrated in the following way:

Suppose a body to move in a straight groove cut in a board while the board moves in a straight line on a table.

Let the body start from A. Let AD be the distance on the board through which the body moves uniformly in a second. Let AB be taken equal and parallel to the distance through which the board moves on the table in a second. And suppose these two motions to take place simultaneously.

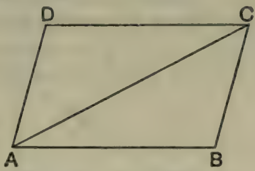

FIG. 147.—Composition of Velocities.

Draw the parallelogram $ABCD$ and its diagonal AC.

Now at the end of a second the position of the body will clearly be the same as if the motions had taken place consecutively instead of simultaneously.

Suppose the board to move through the distance AB first. This brings the body to B. Now let the body move on the board. Then since BC is equal and parallel to AD this motion brings the body to C.

Again, since the two motions take place uniformly and at the same time it is clear that at the end of any fraction of the second the body will have undergone displacements parallel to AB and AD and equal to the same fraction of AB and AD respectively.

This will bring it to a point on AC at a distance from A equal to the same fraction of AC.

Hence the body moves over the table with a uniform velocity which takes it from A to C in the second.

Again if AB, AD are lines drawn parallel and proportional to the actual independent displacements, AC will be parallel and proportional on the same scale to the resultant displacement. And, as these lines represent the displacements per second, they represent the velocities.

Thus if AB, AD represent in direction and magnitude the two independent velocities of the body the diagonal AC of the parallelogram $ABCD$ represents the resultant.

Again, we may take AB, BC, two sides taken in order of the triangle ABC, to represent the two independent velocities; then AC, the third or closing up side, represents the resultant.

This proposition is called the **Triangle of Velocities.**

EXAMPLE.—A train is travelling at the rate of 24 feet per second, and a person throws a stone out of the window at right angles to the train with a velocity of 10 feet per second. Find the actual velocity of the stone.

Draw AB to denote the velocity of the train, and BC at right angles to it to denote the velocity with which the stone is thrown from the train.

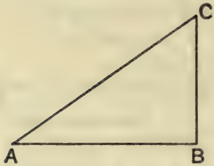

FIG. 148.

Then the actual velocity of the stone is denoted by AC.

$$AC^2 = AB^2 + BC^2.$$

$\therefore$ velocity of stone

$$= \sqrt{(24)^2 + (10)^2} \text{ feet per sec.}$$

$$= \mathbf{26 \text{ feet per sec.}}$$

EXAMPLE.—In what direction must a steamer, which is making 10 knots an hour, head so as to go due east if it is in a current flowing south at the rate of 5 knots an hour?

If AB represents the velocity of the steamer in the current and BC the velocity of the current, AC is the resultant velocity of the steamer.

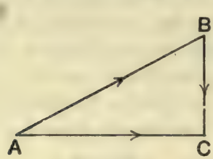

FIG. 149.

Then $AB = 2BC$,

and ACB is a right angle.

$$\therefore \angle BAC = 30°.$$

The steamer must head **30° north of east.**

The same remarks apply to the Triangle of Velocities as to the Triangle of Forces (p. 15). The lines representing the two component velocities are to be drawn in order; one begins where the other leaves off. The resultant velocity is represented by the line drawn from the starting-point of one of these to the stopping-point of the other.

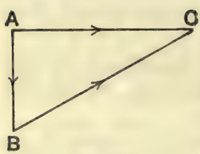

FIG. 150.

In the above example it would be equally correct to draw the figure in a different way. Take AB, BC to represent the velocities of the current and of the steamer, and then AC represents the resultant velocity.

In general the triangle showing the resultant velocity can be drawn in either of two ways.

EXAMPLE.—*AB*, *BC* are two straight lines making the angle *ABC* 120°. A body moves from *A* to *B* with velocity *V*. What velocity must then be given to it, and in what direction, that it may move along *BC* with velocity *V*?

Produce *AB* to *D*, making *BD* equal to *BC*.

Take the lines *BD* and *BC* to represent the velocities *V* in their directions.

Join *DC*.

Then *DC* represents the velocity that must be compounded with *V* along *ABD* to produce *V* along *BC*.

BCD is an equilateral triangle, and *DC* is parallel to the bisector of the angle *ABC*.

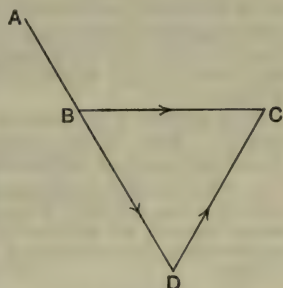

FIG. 151.

Hence the required velocity is *V* **in the direction bisecting the angle** *ABC*.

It is clear that in this question *AB*, *BC* could not have been used to represent the two velocities, because *V* along *AB* is one of the components, and *V* along *BC* is the resultant, and *BC* neither starts where *AB* starts, nor stops where *AB* stops; but *BC* starts where *AB* stops, whereas the resultant must start where one component starts and stop where the other stops.

A different construction could have been made by drawing *CE* parallel to *BA* and making *CE* and *CB* equal, and taking *EC* and *BC* to represent the velocities *V* along *AB* and *BC*. The required velocity would then be represented by *BE*, which would give the same result as before.

Composition of Accelerations.—In just the same manner as for velocities, if two independent accelerations are represented in direction and magnitude by the two straight lines *AB*, *AD* (Fig. 147), their resultant is represented by the diagonal *AC*.

For the accelerations are the velocities acquired per unit of time, and since the two independent velocities acquired per unit of time are AB and AD, the resultant velocity acquired per unit of time is, by the rule for compounding velocities, AC. That is, the acceleration is AC.

Exercises XIX. a.

1. Find the resultant of two velocities, in directions at right angles, of 24 and 7 feet per second.

2. A stone is thrown out of a carriage window with a velocity equal to twice that of the carriage. How must it be thrown that it may move at right angles to the direction in which the carriage is going?

3. A steamer is in a current flowing with half the velocity of the steamer. What is the greatest angle by which the current can divert the course of the steamer?

4. The direction in which a steamer is heading, the direction of the current in which it is, and the magnitude and direction of its actual velocity are all known. Show how to find by a geometrical construction the velocities of the steamer in the water and of the current.

5. A boat starting from one bank of a river and steaming with a known velocity has to proceed straight to a point on the other bank. The rate of the current is known. Give a geometrical construction to show the direction in which the boat must head.

6. A body moves with a uniform velocity V round the perimeter of an equilateral triangle. What velocity must be impressed on it at each angular point?

7. Find by construction the least velocity a boat can have to reach a point on the opposite bank of a river lower down than the one from which it starts, when the stream is flowing at a given rate. And show that with this least velocity it must head in a direction at right angles to that in which it has to go.

8. Show that with any velocity greater than that of the stream a boat can reach any point on either bank.

Independence of Motions.—A body may have two velocities in two different directions, one or both of these velocities being accelerated in these directions, and the two motions will be quite independent of each other.

Frequent examples of this arise from cases of bodies which are projected in directions which are inclined to the vertical. A body projected in this way will have a horizontal component of velocity which remains constant throughout the motion and a vertical component which has the downward acceleration due to gravity.

In questions of this sort the consideration of the vertical motion will generally give us the time for which the body is moving, that is, for instance, before it strikes the ground; and the horizontal motion determines the horizontal distance travelled in that time.

EXAMPLE.—A body is projected with a velocity V at an angle of 45° to the horizon. Show that it will strike the ground at a horizontal distance $\frac{V^2}{g}$ from its point of projection.

Initial vertical component of velocity $=\frac{V}{\sqrt{2}}$.

If body returns to the ground in time t, by

$$s=ut+\tfrac{1}{2}at^2,$$

$$0=\frac{V}{\sqrt{2}}t-\tfrac{1}{2}gt^2,$$

$$\therefore\ t=\frac{\sqrt{2}\,V}{g}.$$

Constant horizontal component of velocity $=\frac{V}{\sqrt{2}}$.

$\therefore$ Distance travelled horizontally in time $\frac{\sqrt{2}\,V}{g}$ is

$$\frac{\sqrt{2}\,V}{g}\cdot\frac{V}{\sqrt{2}}=\frac{V^2}{g}.$$

Exercises XIX. b.

1. A body is projected horizontally with a velocity 60 feet per second from a height of 96 feet above the ground. At what distance, measured horizontally, from its point of projection will it strike the ground?

2. With what velocity must a stone be projected horizontally from the top of a tower 200 feet high so as to reach a point 800 feet from the foot of the tower?

3. Show that the velocity with which a body must be projected in a direction making an angle 60° with the horizon so as to reach a

point on the horizontal plane at a distance a from the point of projection is $\sqrt{\frac{2ga}{\sqrt{3}}}$.

4. Show that a body thrown vertically upwards from a carriage moving with uniform velocity in a straight line (that is, vertically with reference to the carriage) will return to the point in the carriage from which it was thrown.

Uniform Circular Motion.—Suppose a body to move round the circumference of a circle with a velocity which is constant in magnitude and equal to v units of length per unit of time.

In any interval of time the body describes an angle with respect to the centre which is proportional to the time, that is, it passes over an arc of the circle which subtends an angle at the centre proportional to the time.

Circular Measure of Angles.—Now, suppose the angle to be measured in *circular measure.*

The circular measure of an angle, such as the angle AOB, which the arc AB subtends at the centre O of the circle AB, is equal to the arc AB divided by the radius OA of the circle.

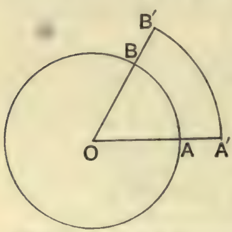

FIG. 152.—Circular measure of angle.

The same value would be obtained whatever be the size of the circle described with its centre at the angular point O, and having an arc lying within the angle, because with a given angle the arc is proportional to the radius.

If we take a larger circle $A'B'$ with centre O from which to determine the angular measure of the angle AOB, we would get in one case

$$\frac{AB}{OA} \text{ and in the other } \frac{A'B'}{OA'};$$

and these two are equal to one another.

Hence the measure *arc÷radius* is always the same for a given angle, no matter what the size of the radius taken.

Thus we have the definition,

$$\textit{circular measure} = \frac{\textit{arc}}{\textit{radius}}.$$

Angular Velocity.—The rate at which a body or point moving round the circle increases the angle described with reference to the centre is called its *angular velocity*.

Denote the radius OA by r.

Thus, if the arc AB is described in time t;

$$\textit{linear velocity of moving point} = AB \div t;$$

$$\textit{angular velocity of moving point} = \frac{AB}{r} \div t.$$

Let the angular velocity be denoted by ω.

Then $$v = \frac{AB}{t}; \quad \omega = \frac{AB}{rt}.$$

$$\therefore \ \omega = \frac{v}{r}; \text{ or } v = \omega r.$$

We may show this in another way.

By definition,

$$\omega = \text{angle described in unit of time}$$

$$= \frac{\text{arc described in unit of time}}{\text{radius}}$$

$$= \frac{v}{r} \ (\because v \text{ means arc described in unit of time}).$$

$$\therefore \ v = \omega r.$$

This is a very important result.

Speed.—The word *speed* is often used to denote the mere magnitude of a velocity without any reference to its direction.

To specify a velocity we must state both magnitude and direction. Speed means magnitude of velocity only. In other words, a velocity is a speed in a definite direction.

It will be convenient to speak of the speed of a body moving in a circle, the body having a velocity that is constantly changing in direction.

The angle whose circular measure is unity, or whose arc is equal to the radius of the circle, is called a *radian*.

In many questions on circular motion account has to be taken of the circular measure of the angle described by a body which moves once round the circumference of a circle.

The measure of this angle is

$$\frac{\text{circumference of circle}}{\text{radius of circle}}.$$

The ratio of the circumference of a circle to its diameter is usually denoted by the symbol π; and is numerically equal to about $3\frac{1}{7}$, or more exactly $3{\cdot}1416$.

Thus, if r is the radius of a circle, circumference $= 2\pi r$.

Circular measure of angle described by body which moves once round circumference is

$$\frac{2\pi r}{r} = 2\pi.$$

In other words we may say that the circular measure of four right angles is 2π; or, again, that four right angles are equal to 2π radians.

If a body goes uniformly round the circle n times in a second, its angular velocity is

$$2\pi n,$$

for this is the entire angle swept out in a second.

Geometrical Propositions.—Before considering the Dynamics of a body or heavy particle moving in a circle it will be necessary to notice the following geometrical propositions:

Let AB be a small arc of a circle of given radius OA.

Draw BN perpendicular to OA, and the tangent BT to meet OA produced at T.

Then BA, the arc, is intermediate in length between BN and BT.

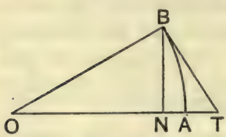

FIG. 153.

For BN being the perpendicular from B on OT is shorter than any other path from B to OT.

And BA is, on the whole, a more direct path from B to OT than BT is; that is, the direction of BA at any point more nearly coincides with the direction of a perpendicular to OT than that of BT does; except only at the point B, where the directions of BA and BT coincide with each other.

Hence BA is shorter than BT.

[This reasoning, although not professing to be mathematically rigid, shows, in a general way, that BA is longer than BN and shorter than BT.]

Now suppose that while OA remains of the same length, AB becomes indefinitely short.

Then $$\frac{BN}{BT} = \frac{ON}{OB}.$$

But $OB = OA$; and when B is indefinitely near to A, ON becomes equal to OA;

$$\therefore \frac{BN}{BT} = 1, \text{ when } AB \text{ is indefinitely short.}$$

And BA is greater than BN and less than BT.

$$\therefore \frac{BN}{BA} = 1, \text{ when } AB \text{ is indefinitely short.}$$

Again, $$\frac{NT}{BN} = \frac{NB}{ON};$$

and $$\frac{NB}{ON} = 0, \text{ when } AB \text{ is indefinitely short;}$$

$$\therefore \frac{NT}{BN} = 0, \text{ in the same case.}$$

And NA is less than NT, and $\frac{BN}{BA} = 1$ ultimately.

$$\therefore \frac{NA}{BA} = 0, \text{ when } AB \text{ is indefinitely short.}$$

The results to be remembered are:

When the arc AB is taken indefinitely short,

$$\frac{BN}{BA} = 1; \quad \frac{NA}{BA} = 0.$$

We might say that BN and BA become equal. But this does not express all unless we are careful in what sense we are to understand the statement. The quantities become equal, of course, since each becomes zero. But the point to notice is that their ratio ultimately becomes unity, or *their difference vanishes in comparison with either of them.*

With regard to the second result, we may say that NA *vanishes in comparison with* BA.

Acceleration of a Point moving with Uniform Speed in a Circle.—Suppose a point to move with uniform speed v along the circumference of a circle of radius r.

Then, although the speed is constant, the velocity is constantly changing, for it changes in direction. And a change of velocity implies an acceleration. We shall therefore investigate what the acceleration is in this case.

Now acceleration at a given instant means rate of increase of velocity at that instant. We shall find it by considering the mean rate of increase of the velocity in a short interval of time, and then finding the limiting value of this when the time is made indefinitely short.

Since the motion of the body does not keep to a fixed direction, we shall not expect the acceleration at any instant to be in the direction of the motion. We shall examine its two components in, and at right angles to, this direction. The entire acceleration is compounded of these.

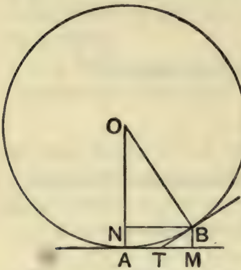

FIG. 154.—Uniform motion in a circle.

Let O be the centre of the circle, and A, B two neighbouring points on the circumference.

Draw the tangents AT, BT at A and B.

Draw BM, BN perpendicular to AT, OA.

At A the point has velocity v along AT.

At B the point has velocity v along TB.

This last may, by the triangle of velocities, be resolved into two components in the directions of TM, MB, and equal to

$$v \cdot \frac{TM}{TB} \text{ and } v \cdot \frac{MB}{TB} \text{ respectively.}$$

Since the triangle TMB is similar to ONB, these are equal to

$$v \cdot \frac{ON}{OB} \text{ and } v \cdot \frac{NB}{OB} \text{ respectively.}$$

Thus, as the point moves from A to B it acquires the following velocities.

In direction of AT the velocity

$$v\frac{ON}{OB} - v = -v \cdot \frac{NA}{OA};$$

In direction of AO the velocity

$$v \cdot \frac{NB}{OB} = v \cdot \frac{NB}{OA}.$$

Now the time in which these increments of velocity are acquired is the time taken by the point to move from A to B, that is $\frac{AB}{v}$.

Hence, as the point moves from A to B, its mean rates of increase of velocity in the directions AT, AO are, respectively,

$$-v \cdot \frac{NA}{OA} \div \frac{AB}{v}, \quad \text{or} \quad -\frac{v^2}{r} \cdot \frac{NA}{AB};$$

and

$$v\frac{NB}{OA} \div \frac{AB}{v}, \quad \text{or} \quad \frac{v^2}{r} \cdot \frac{NB}{AB}.$$

The actual accelerations of the point when it is at A, in the directions AT, AO, are the limiting values of these quantities when AB is made indefinitely short.

Then

$$\frac{NA}{AB}=0, \quad \frac{NB}{AB}=1.$$

Hence, there is no acceleration along AT: *the entire acceleration is along AO and is equal to* $\frac{v^2}{r}$.

Again, if ω is the angular velocity, $v=\omega r$,

$$\therefore \text{ acceleration is } \omega^2 r.$$

Thus, if a is the acceleration, we have the important results,

$$a \text{ is along } AO;$$

$$a=\frac{v^2}{r}=\omega^2 r.$$

Dynamical Result.—Suppose that a body of mass m is constrained to move with uniform velocity v along the circumference of a circle of radius r.

Then, since the body has an acceleration which is towards the centre of the circle, the resultant force acting on it is towards the centre of the circle. For, by the Second Law of Motion change of motion is in the direction in which the force producing it acts.

Again, the acceleration is $\frac{v^2}{r}$.

Therefore the resultant force acting on the body towards the centre of the circle is, by the equation $\mathbf{f}=ma$, equal to

$$\frac{mv^2}{r} \text{ absolute units.}$$

This is also equivalent to $m\omega^2 r$ absolute units.

This is the force which has to be constantly applied to the body to keep it in its circular path. If this force were removed the body could only move in a straight line, according to the First Law of Motion.

As an example of the action of a force in keeping a body moving in a circle, if a stone is tied by a string and whirled round, a tension in the string acts on the stone, constantly pulling it towards the centre, and so keeping it in its circular path. If the string breaks, the stone begins to move off along a straight line, namely, the tangent to the circular path at the point at which it is at the instant when the string breaks.

Centrifugal Force.—The tendency of the body to get further away from the centre of the circle is sometimes looked upon as if it were due to a force pulling it away from the centre equal and opposite to the force required to keep it in the circular path, and this supposed force pulling the body from the centre is called *centrifugal force.* There is, however, no such force acting. A force is required to act on the body towards the centre, because, with the given motion, the body has acceleration towards the centre, not because there is any other force to be overcome.

To make this plainer, imagine a body to be moved along with acceleration in a straight line. A force is necessary to do this, not because there is any force pulling the body back, but because its motion is being accelerated. Whenever a mass undergoes acceleration in any direction, a force must act on it in that direction.

The name 'centrifugal force' is then rather unscientific and misleading, but as it is frequently used, it is well to understand what is meant by it. We may say that it means the tendency of the body to get further away from the centre in consequence of its own inertia. It is supposed to be equal to the force which acts towards the centre to hold the body in its path.

Centripetal Force.—The force acting on the body and directed towards the centre to hold it in its path is sometimes called the *centripetal force.*

EXAMPLE.—With what velocity is a point on the circumference of a circle moving, the circle having a diameter of

10 inches and rotating about its axis uniformly 500 times in a minute?

In one second circle rotates $\frac{500}{60}$ times $=\frac{25}{3}$ times.

$\therefore$ angular velocity $=\frac{25}{3}\times 2\pi$

$$=\frac{50\pi}{3} \text{ radians per second.}$$

Radius of circle $=\frac{5}{12}$ foot.

$\therefore$ velocity of point on circumference

$$=\frac{50\pi}{3}\cdot\frac{5}{12} \text{ feet per second}$$

$$=\frac{250\times 22}{36\times 7} \text{ feet per second}$$

$$=\frac{1375}{63} \text{ feet per second}$$

$$=\mathbf{21\tfrac{52}{63}} \textbf{ feet per second.}$$

Example.—What fraction of the weight of a body is required to keep it from flying off the earth's surface at the equator, supposing the equatorial radius of the earth to be 4000 miles and that it makes one complete revolution on its axis in a solar day of 24 hours?

Angular velocity of earth is

$$\frac{2\pi}{24\times 3600}=\frac{44}{7\times 24\times 3600}=\frac{11}{7\,.\,24\,.\,900} \text{ radians per second.}$$

Consider a body of mass m lbs.

Force required to keep this in its circular path is towards the centre of the earth and equal to

$$m\left(\frac{11}{7\,.\,24\,.\,900}\right)^2\times 4000\times 5280 \text{ poundals}$$

$$=m\,.\,\frac{11^3}{7^2\,.\,3\,.\,9^2} \text{ poundals.}$$

Weight of body $=32\ m$ poundals.

$\therefore$ ratio of force required to weight

$$=\frac{11^3}{7^2\,.\,3\,.\,9^2\,.\,32}=\frac{1}{286} \textbf{ about.}$$

Exercises XIX. c.

1. Find in radians per second the angular velocity of the hour hand of a clock.

2. Show that the extremity of the minute hand of a clock, 16 feet long, passes over 1 inch in about 3 seconds.

3. Find the number of miles passed over in space by the earth in 1 second in consequence of its annual rotation round the sun, supposing its path to be a circle of 90 million miles radius.

4. A stone weighing half a pound is tied to the end of a string 3 feet long and whirled round twice a second. What is the tension in the string in pounds' weight?

5. A body of mass 2 lbs. is fastened to the rim of a wheel, and when the wheel rotates 8 times a second the force necessary to hold the body on is 121 lbs.' weight. What is the radius of the wheel?

6. Show that the attraction exercised by the sun on the earth to keep it in its path, supposing this to be a circle of 90,000,000 miles radius, and that the earth travels round the sun once in 365 days is equal to about ·00059 of a pounds' weight per pound mass of the earth.

7. Show that if the earth rotated about its axis 17 times more quickly than it does, the weight of a body would not cause it to rest on the surface at the equator.

8. What is the rule for finding the magnitude and direction of the acceleration which is produced by compounding together two given accelerations? What is the joint effect of an acceleration northward that would be produced by a force of 50 dynes when acting on 2 grams, and of an acceleration eastward such as would in three seconds give a body initially at rest a velocity of 21 centimetres per second? (London Matric., 1898.)

9. A particle is moving in a circle whose radius is 8 feet, and the force which keeps it moving in the circle equals the weight of the body at a place where $g=32$; find the velocity of the particle.

State what is meant when it is said that at a certain place g is equal to 32. (Science and Art, 1897.)

10. A particle moves in a circle whose radius is r, and the force required to keep it moving in the circle is F, along what line and in what direction does F act? Also show that the kinetic energy of the particle is $Fr \div 2$. (Science and Art, 1897.)

11. A mass of one pound is whirled uniformly at the end of a string 2 feet long; it moves three times round in $1\frac{1}{5}$ seconds. Express the tension of the string in poundals. (Science and Art, 1898.)

Summary.

Relative Velocity and Acceleration.—If two bodies A and B are moving along the same straight line, then the rate at which A's distance from B is increasing is called A's velocity relative to B, and the rate at which this relative velocity is increasing is called A's acceleration relative to B.

If A's velocity and acceleration are v and a; and B's are v' and a', then whatever be the relative positions of A and B and the magnitudes of v, a, v', a', A's velocity and acceleration relative to B are $v - v'$ and $a - a'$.

Composition of Velocities.—A body may have at the same time two independent velocities. These give rise to a single resultant velocity.

If the two component velocities are represented by two sides of a parallelogram drawn from a point, the resultant is represented by the diagonal through the same point.

If the two component velocities are represented by two sides of a triangle taken in order, the resultant is represented by the third side.

Composition of Accelerations.—This is done in the same way as for velocities.

Uniform Circular Motion.—The circular measure of an angle is arc ÷ radius.

If a body moves round the circumference of a circle, the rate at which it describes an angle (measured in circular measure) about the centre is called its angular velocity.

If r is radius of circle,
ω angular velocity,
v linear velocity;

then
$$\text{angular velocity} = \frac{\text{distance per sec.}}{\text{radius}},$$

i.e.
$$\omega = \frac{v}{r},$$

or
$$v = \omega r.$$

Speed means the magnitude of a velocity without reference to its direction.

The ratio of circumference of circle to radius is denoted by π (which is about $3\frac{1}{7}$).

Hence the circular measure of four right angles is 2π.

If a body goes round circle n times per second, its angular velocity is $2\pi n$.

If a body moves with uniform speed v round a circle of radius r, its acceleration at any instant is to the centre, and is equal to v^2/r or $r\omega^2$.

Resultant force acting on body is mv^2/r or $mr\omega^2$.

CHAPTER XX.

SIMPLE HARMONIC MOTION. PENDULUMS.

Simple Pendulum.—A simple pendulum is a small heavy body suspended at the end of a light string.

The words 'small' and 'light' denote that the *size* of the body and the *mass* of the string are negligible.

The body is sometimes called the **bob** of the pendulum.

If the pendulum is pulled on one side and then left to itself it will swing backwards and forwards, making oscillations.

Meaning of Oscillation.—By an oscillation is meant a **to-and-fro swing**, the motion from one end of the path to the other and back again.

In practice, when a pendulum is set to swing and left to itself the oscillations gradually become smaller and smaller, and it at length comes to rest. This is due to the resistance of the air and to the small amount of resistance to bending that the string offers. If there were no resistances of this sort the oscillations would not diminish in size.

Energy changes during an Oscillation.—Suppose these disturbing causes not to exist. Then when the bob is at the extremity of its path its velocity is for an instant zero. It is then in its highest position. So that it has lost all its kinetic energy and possesses the maximum of potential energy. As it descends its velocity increases, and becomes a maximum when it is in its lowest position, or when the string is vertical. It ascends to an equal height on the other side coming to rest for an instant at the extremity of its path, and so on, the sum of its potential and kinetic energies remaining constant.

The motion of the pendulum will be better understood when we have studied the characteristics of the sort of motion of which it forms an example.

Simple Harmonic Motion.—Suppose a point P to move with uniform speed round the circumference of a circle with centre O.

Let Q be the foot of the perpendicular from P on the fixed diameter AOA'.

As P moves round the circle Q moves to-and-fro along AA'; and while P, starting from A, moves round the circle once, Q moves from A to A' and back to A.

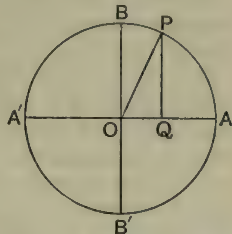

FIG. 155.—Simple harmonic motion.

The motion of Q in the path AA' is called *Simple Harmonic Motion.*

This is a very important case of motion in Mechanics and Physics; it is written, for short, S.H.M.

The velocity of Q at any instant is the resolved part of P's velocity in the direction of AA'.

Draw the diameter BOB' at right angles to AOA'.

Suppose P to move in the sense $ABA'B'A$.

When P is at A or A', Q is at A or A', and the velocity of Q is then zero.

When P is at B or B', Q is at O; then the speed of Q is a maximum, being equal to that of P.

Thus the motion of Q is such that at A it has for an instant zero velocity; it moves towards O with velocity which increases till it reaches O; then the velocity is a maximum; it then decreases till Q reaches A', when the velocity is again zero. As Q moves from A' back to O the velocity undergoes the same changes in magnitude as before; and so on.

It is more important, however, to consider the acceleration of Q at any point in its path.

The acceleration of Q is always in the line AA'; for since its component of velocity at right angles to AA' never changes from zero it has no component of acceleration at right angles to AA'.

The acceleration of Q is the component along AA' of the acceleration of P.

Now let $OA = R$; and suppose that the angular velocity of P round the circle is ω.

P's acceleration is $\omega^2 PO$ along PO.

This may be resolved into $\omega^2 PQ$ and $\omega^2 QO$ in the directions PQ, QO.

$\therefore$ the acceleration of Q is $\omega^2 QO$ along QO.

Whatever the position of P, and whether Q is moving towards or from O, the acceleration of Q is $\omega^2 QO$ and is from Q to O.

It is easily seen that the acceleration of Q is always towards O, because when Q is approaching O its velocity is increasing in magnitude, and when Q is receding from O its velocity is decreasing in magnitude.

If a point Q moves in a straight line AA' in such a manner that its acceleration is directed to a fixed point O in AA' and is proportional to QO, then the motion of Q is S.H.M. (This is the converse of what has just been shown.)

For, suppose the acceleration of Q to be $k \,.\, QO$.

And let the velocity of Q at the point A be zero, so that Q starts from A towards O.

Now, suppose a point P to start from A at the instant when Q starts from A, and to move with angular velocity $\sqrt{k}$ round a circle with centre O and radius OA.

Let K be the foot of the perpendicular from P on AA'. The acceleration of K is $\sqrt{k} \,.\, KO$ towards O.

And since Q starts from A with K, each having the same initial velocity, zero, and each having the same acceleration at each point of the path, the velocity and the position of each must always be the same. That is, Q always coincides with K.

But the motion of K is S.H.M.

$\therefore$ the motion of Q is S.H.M.

The time that P takes to travel once round the circle with angular velocity ω is $\frac{2\pi}{\omega}$.

This, therefore, is also the time that Q takes to move from A to A' and back again. This time is called the *period* of the S.H.M.

If the angular velocity of P is ω, the acceleration of Q is

$\omega^2 . QO$. So that if Q has acceleration in any position $k . QO$ the corresponding angular velocity of P is $\sqrt{k}$.

The period of Q's motion is therefore $\frac{2\pi}{\sqrt{k}}$.

Motion of Pendulum.—Let SO be the string of the pendulum, S being the point of suspension and O the position of the bob when not swinging. Let $SO = l$.

Suppose the pendulum is set to oscillate in the path AOA'.

Let P be the position of the bob at any instant.

If m is its mass, it is acted on by a force mg vertically downwards.

Draw PT the tangent at P.

The force mg is equivalent to

$$mg . \frac{SP}{ST} \text{ and } mg\frac{PT}{ST}$$

in the directions of SP and PT.

Now the motion at P is in the direction of PT; and it is the force $mg\frac{PT}{ST}$ which produces the acceleration in this direction.

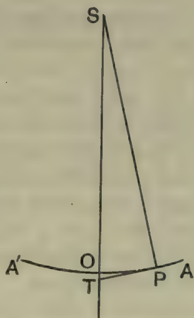

FIG. 156.—Motion of simple pendulum.

$$\therefore \text{ the acceleration along } PT \text{ is } g . \frac{PT}{ST}.$$

Now suppose that the oscillations are very small, so that even in the extreme positions A, A' the string is deviated only a very little from the vertical.

Then the path AOA' of P becomes nearly a straight line; and in the expression for the acceleration at P, PT becomes practically equal to PO and ST to SO, that is, to l; so that the acceleration is $\frac{g}{l} . PO$.

The motion of P is then approximately the same as the motion of a point in a straight line AOA' moving with an acceleration that is at every instant directed toward O and is equal to $\frac{g}{l} . PO$.

Thus when the pendulum makes *small oscillations* its motion is approximately a S.H.M.

The period of the motion is $2\pi \div \sqrt{\frac{g}{l}}$, that is, $2\pi\sqrt{\frac{l}{g}}$.

This is an important expression and should be remembered. Note that it means the time taken by the pendulum to swing from one end of its path to the other *and back again.* It is also called the *time of an oscillation.*

Sometimes the half swing, from A to A', is called an oscillation ; but it is better to use the word to mean the motion of the pendulum up to the instant when it gets back to its original state, that is, the motion from A to A' and back to A.

Determination of g by Simple Pendulum.—The simple pendulum affords an accurate method of determining the acceleration due to gravity.

A small bob is suspended by a very fine string, and the distance from the point of suspension to the centre of the bob is measured. Call this l. The pendulum so formed is set to make small oscillations. By observing the time of several oscillations, and dividing this by the number of oscillations, the time of a single oscillation is found with considerable accuracy. Let this be T.

Then we have

$$T=2\pi\sqrt{\frac{l}{g}}.$$

$$\therefore\ g=\frac{4\pi^2 l}{T^2}.$$

Thus g can be calculated from quantities that are known.

Unavoidable Errors.—Several small errors are unavoidable in the use of the simple pendulum. The bob is of a finite size and is not an infinitesimally small particle, as has been supposed: the string though very light has some mass : the string is not infinitely flexible, but some force is required to bend it. All these circumstances influence to some extent the time of an oscillation and introduce errors in the value of g as calculated by the formula given above.

Compound Pendulum.—A rigid body such as a rod or bar, set to oscillate under the action of gravity about an axis rigidly connected with it, is called a compound pendulum.

Suppose the figure to represent a compound pendulum. Let G be its Centre of Gravity, and take the plane of the figure, or the plane of the paper, to be the vertical plane passing through G.

Let the axis about which the pendulum oscillates be a horizontal axis at S. This axis may be supposed to be at right angles to the plane of the figure.

Then, when the pendulum is at rest, SG is a vertical straight line.

The point S which is in the axis, and vertically above the C.G. of the pendulum when the pendulum is at rest, is called the *Centre of Suspension* of the pendulum.

FIG. 157.—Compound pendulum.

The pendulum will oscillate in the same time as a theoretical simple pendulum of a certain length.

Such a simple pendulum is called the *simple equivalent pendulum* of the given compound pendulum.

Let SG be produced vertically downwards to the point O, so that SO is equal to the length of the equivalent simple pendulum, or so that a particle suspended at O by a weightless perfectly flexible string SO, would oscillate in the same time as the given compound pendulum.

Then the point O is called the *Centre of Oscillation* of the given pendulum.

Suppose the pendulum to be provided with an axis at O parallel to the axis at S. Let it be turned upside down and set to oscillate about O. Then the theory of the compound pendulum shows that it will oscillate in just the same time as it does about the axis at S. That is, when the old centre of oscillation is made the centre of suspension, the old centre of suspension becomes the centre of oscillation. In other words, *the centres of suspension and oscillation are convertible.*

It must be noticed that in this the two axes S and O must be parallel to each other.

Suppose that for a compound pendulum positions for the axes at S and O have been found by trial, which make the times of oscillation about them exactly equal. Let the distance SO be accurately measured, and be l. Let the time of oscillation about either axis be T. Then SO is the length of the simple equivalent pendulum of the compound pendulum for oscillation about either S or O. Thus T is the time of oscillation of a simple pendulum of length l. And we have the relation

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

This experiment affords the best method known of determining the value of g.

Precautions in an Accurate Experiment.—Further refinements are introduced into the experiment in practice. The resistance of the air affects the time of the oscillation by opposing the motion, and the upward buoyancy of the air on the pendulum also affects the time. These sources of error are avoided by causing the pendulum to oscillate in a vacuum.

The time of an oscillation, even if the oscillation is small, is not quite the same as if it were infinitesimally small, that is, it depends on the size of the oscillation to some extent. The size of the oscillation is therefore taken into account in accurate experiments.

In order to avoid friction at the axis as far as possible, and, at the same time, to have a definite mathematical straight line about which the oscillation is taking place, each axis is made of the form known as a *knife-edge*; that is, the axis is of the shape of a prism with a triangular cross-section, as shown in the figure, the lowest edge resting on two hard horizontal planes, one on each side of the pendulum, when the oscillation is about that axis.

Numerous determinations of the value of g have been made with a compound pendulum by Captain Kater; and the pendulum, designed for use in this manner, is frequently called *Kater's pendulum.*

Effect of Earth's Rotation and Shape.—It should be noticed that even when a body is at rest on the surface of the earth, that is, not moving relatively to the earth, it has an acceleration towards the earth's centre in consequence of the earth's rotation.

Now if a body is falling freely, that is, moving under the action of its own weight, or the attraction of the earth on it, it has an entire resultant acceleration, which is made up of two:

(i) Its apparent acceleration or the acceleration which it seems to have on the supposition that the earth is at rest;

(ii) An acceleration due to the earth's rotation, that is, the acceleration that it would have if not moving with respect to the earth.

In experiments with Attwood's machine, or a pendulum, it is the apparent acceleration that is found. If we compound with this the acceleration which the body is known to possess in consequence of the earth's rotation we get the true acceleration.

Again the true weight of a body is the earth's attraction on it. But this is slightly different from what may be called, in this connexion, the apparent weight, which is the force that would be necessary to produce the apparent acceleration in it.

The apparent acceleration differs from the true acceleration, as a rule, both in direction and magnitude. The differences are only very slight. The same applies to the apparent and true weights. They may, in general, be taken to be the same. The difference between them is greatest at the equator, where it is about $\frac{1}{290}$ of either, and where they coincide in direction. To be precise, however, it should be remembered that g is the *apparent* acceleration due to gravity.

Thus g varies from point to point of the earth's surface in consequence of the earth's rotation. But there is another reason for the variation of g.

The actual force of attraction which the earth exerts on a body on various parts of its surface varies, because of the shape of the earth, which is not truly spherical. The earth is flatter at the poles than at the equator; it bulges slightly at the equator. Consequently a body moved along its surface from the equator towards a pole gradually becomes slightly nearer to the earth's centre, and, in consequence, the earth's attraction on it slightly increases.

For both these reasons the value of g (or the weight of a given body) is least at the equator, and gradually increases towards the poles, where it is greatest.

Values of g in different localities on the earth's surface.

	In feet and seconds.	In centimetres and seconds.
At equator, - -	32·09	978·1
In latitude 45°, -	32·17	980·6
At pole, - - -	32·25	983·1

Variation in Weight of a Body.—The weight of a body also depends to a slight extent on its elevation. If it is carried up a mountain it is carried further away from the earth's centre, and its weight falls off. If the body is carried down into a mine, then, although it is carried nearer to the earth's centre, some parts of the earth's substance attract it upwards, and its weight falls off. So that the weight is greater *at* the surface than either above or below it.

Thus, on the whole, the weight of a body depends on *latitude* and *altitude.*

When a body is weighed in a balance, or with a weighing machine or steelyard, in order to determine its mass, the same result will be obtained no matter in what locality the weighing is done. For, although the weight of the body varies, the weight of the weights used varies in exactly the same manner: so that if certain weights balance the body in one locality they will do so in any other locality.

If, however, the body is weighed by means of a spring-balance, or any such instrument, in which the indication is given by the deformation produced in a spring or other body by means of the force acting on it, then this indication will depend on the actual weight of the body in the given locality, and will, therefore, vary to some extent with the locality.

Thus, when a spring-balance is used, a body will appear to weigh more at the poles than at the equator, and will appear to weigh more at the surface of the earth than when carried up a mountain or down a mine.

Thus if, for example, a spring-balance is correctly graduated for use in London, a body weighed with it at the equator will appear to weigh too little, and a body weighed with it at the pole will appear to weigh too much.

Since the extreme variation of the weight of a body at various parts of the earth's surface is nearly as much as ·2 in 32, it follows that the mass of a body as estimated by means of a spring-balance may vary by nearly as much as $\frac{1}{10}$ of an ounce per pound.

Seconds' Pendulum.—A *seconds' pendulum* for a given locality means a pendulum which will make one beat, or half of a complete oscillation, in each second, when making very small oscillations.

The length of the seconds' pendulum is the length of the theoretical simple pendulum which makes one beat per second, when making very small oscillations.

Since the time of a complete small oscillation is given by the formula

$$T = 2\pi\sqrt{\frac{l}{g}},$$

it follows that if l is the length of the seconds' pendulum at a place where the apparent acceleration due to gravity is g,

$$1 = \pi\sqrt{\frac{l}{g}}.$$

$$\therefore \quad \pi^2 l = g.$$

Thus, the length of the seconds' pendulum varies from point to point of the earth's surface, and is directly proportional to g.

Again, it follows from the formula for the time of a complete oscillation that the length of the pendulum which oscillates in any given time must be proportional to the value of g at the locality in which it is. Thus, if a clock is regulated to keep correct time in a given locality, it will not, as a rule, do so in another locality, but to regulate it the length of its pendulum must be altered.

EXPERIMENT 44.—Make a simple pendulum by attaching a small weight to the end of a fine string: a leaden bullet of about half an inch diameter with a hook driven into it for the string will do very well. Fix a long nail in the wall so that it projects a couple of inches. Hang the pendulum to this nail so that the bob swings quite clear of the wall; and make a vertical mark on the wall behind the position of the bob and string when they are at rest.

Make the string about 1 metre long: measure the length from the point of suspension at the nail to the middle of the bob, and take this for the length of the pendulum.

Set the pendulum swinging by drawing the bob aside about 2 inches and then releasing it, taking care that it swings in a vertical plane and has nothing like a circular motion. Count the time of a complete oscillation as the time between an instant when the pendulum crosses the vertical mark moving from left to right to the next instant when it crosses the mark moving the same way. Observe the time taken to make a considerable number of oscillations, say about 20, and thus calculate the time of one oscillation.

Take various lengths for the pendulum, say 50, 60, 80, 100 centimetres. Determine the time of an oscillation as before. Draw up a table of the squares of the times. These should be proportional to the lengths; or when divided by the lengths should give the same number.

Determine the value of g in centimetres and seconds by means of the equation

$$T=2\pi\sqrt{\frac{l}{g}},$$

from which $$g=\frac{4\pi^2 l}{T^2}.$$

The same experiment may, of course, be done using feet to measure the lengths instead of centimetres, and the result will be in feet and seconds.

Or, if the value of g is found in one measure, it can easily be converted into the other. For since a foot is 30·48 cms., an acceleration of 1 foot per second per second is an acceleration of 30·48 cms. per second per second, and to convert from British to metric units we have only to multiply by 30·48, and to convert from metric to British units to divide by the same number.

EXAMPLE.—Find the length of the pendulum which will make 24 oscillations in a minute.

Let the length be l feet.

The time of an oscillation $\frac{60}{24}$ secs. $=\frac{5}{2}$ secs.

By formula $$T=2\pi\sqrt{\frac{l}{g}}.$$

$$l=\frac{gT^2}{4\pi^2}=\frac{32\,.\,25\,.\,49}{4\,.\,4\,.\,484}$$

$$=5\tfrac{15}{242}.$$

The required length is $5\frac{15}{242}$ **feet.**

EXAMPLE.—A pendulum beats seconds. If it is lengthened by $\frac{1}{100}$ inch show that in one day it will lose about 11·2 beats.

Let l feet be the original length.

Then $$1=\pi\sqrt{\frac{l}{g}}.$$

$$\therefore\ l=\frac{g}{\pi^2}.$$

When the pendulum is lengthened the time of a beat is

$$\pi\sqrt{\frac{l+\frac{1}{1200}}{g}}=\pi\sqrt{\frac{\frac{g}{\pi^2}+\frac{1}{1200}}{g}}=\sqrt{1+\frac{\pi^2}{1200g}}$$

$$=\sqrt{1+\frac{484}{49\,.\,1200\,.\,32}}$$
$$=\sqrt{1\cdot00026}$$
$$=1\cdot00013 \text{ nearly.}$$

Number of seconds in one day is

$$3600\times24=86400.$$

$\therefore$ number of beats made in one day is

$$\frac{86400}{1\cdot00013}=86388\cdot8 \text{ about.}$$

$\therefore$ number of beats lost in a day $=$ **11·2 about.**

Exercises XX.

1. If the length of the seconds' pendulum is 39·14 inches, what is the length of the pendulum that will make a complete oscillation in one second?

2. If a pendulum is taken from a place whose latitude is 45° to the pole, show that in order that it may oscillate in the same time as before it must be lengthened by about $\frac{1}{400}$ of its original length.

3. Show that the ratio of the times of oscillation of a pendulum at the equator and at the pole is about 401 : 400.

4. If a clock is regulated to keep correct time at the equator, find how much it will gain in a day at the pole.

5. What will a hundredweight appear to weigh at the equator if weighed with a spring balance which is graduated for use in London, the values of g at the equator and in London being 32·09 and 32·19?

6. If a London spring-balance is used at the pole, show that the indications must be corrected by subtracting $\frac{8}{43}$ per cent.

7. When the time of oscillation of a simple pendulum is given by the formula $$2\pi\sqrt{\frac{l}{g}},$$
what is the meaning of the word "oscillation"?

Define a compound pendulum. If such a pendulum makes 25 oscillations in a minute, what is the distance from the centre of suspension to the centre of oscillation? (Science and Art, 1898.)

8. Define a simple pendulum and an oscillation of a pendulum. The formula for the time of a small oscillation being

$$2\pi\sqrt{\frac{l}{g}},$$

state exactly what is meant by π, l, and g.

What is a seconds' pendulum? If we could have a simple pendulum in a place where $g=30$ in feet and seconds, what would be the length of a seconds' pendulum? (Science and Art, 1897.)

Summary.

A **simple pendulum** is a small heavy body suspended at the end of a light string.

An **oscillation** of a pendulum means the motion from one end of its path to the other and back again.

Simple Harmonic Motion.—If a point P moves with uniform speed round the circumference of a circle, and Q is the foot of the perpendicular from P on a fixed diameter of the circle, the motion of Q along this diameter is called Simple Harmonic Motion, or S.H.M.

In S.H.M. the moving point has at any instant an acceleration directed to its mean position, and proportional to its distance from this position.

Conversely, if a point moves in a straight line, and has at each instant an acceleration proportional to its distance from a fixed point in the line, and towards this point, it moves with S.H.M.

If the acceleration is $k \times$ displacement from mean position, the period of the motion is $2\pi/\sqrt{k}$.

If a simple pendulum of length l makes small oscillations, the period or time of a complete to-and-fro swing is

$$2\pi\sqrt{\frac{l}{g}}.$$

The value of g can be found with considerable accuracy by finding the time of oscillation of a simple pendulum of known length.

Compound Pendulum.—This is a rigid body, generally a bar, which can oscillate about a horizontal axis.

The **centre of suspension** is the point of this axis that is vertically above the equilibrium position of the C.G. of the pendulum.

The **centre of oscillation** is the point in the pendulum vertically below the C.G. when the pendulum is at rest, and at a distance from the centre of suspension equal to the length of a theoretical simple pendulum that will oscillate in the same time.

This simple pendulum is called the **equivalent simple pendulum** for the given compound pendulum.

The centres of suspension and oscillation are convertible.

If l is the distance between the centres S and O, and T is the time of oscillation about either, then T is period for simple pendulum of length l.

$$\therefore \quad T = 2\pi\sqrt{\frac{l}{g}}.$$

This gives the best known method of finding g.

The compound pendulum used to determine g is called Kater's pendulum.

The value of g, the apparent acceleration of a freely falling body (as well as the weight of a given body), depends to a slight extent on the rotation and on the shape of the earth. It is least at the equator and greatest at the poles.

g also depends on the height above or depth below the earth's surface, being greatest in a given locality *at* the earth's surface.

If a body is weighed with a balance, or such apparatus, against standard weights, the result is independent of the locality, because weights of body and of weights vary in the same ratio from one locality to another. But if a body is weighed with a spring balance, the indication, depending on the weight of the body, varies with the locality to a slight extent.

A **seconds' pendulum** is one which makes one beat, or one half oscillation, in a second. Its length depends on the locality, and is given by the equation $\pi^2 l = g$.

EXAMINATION PAPERS.

BOARD OF EDUCATION, SOUTH KENSINGTON (1900).

Theoretical Mechanics: Solids.

(You are not permitted to answer more than *seven* questions.)

1. Given 1 metre = 39·37079 in. and 1 kilogramme = 2·20462 lbs., express a force of 1·033 kgs. per square centimetre in pounds per square inch. (10)

2. How many things are involved in the specification of a force?

Forces, 17 and 19 lbs. respectively, act at a point and the angle between them is 60°; construct on a scale of 1 lb. to $\frac{1}{8}$-inch the parallelogram of forces and determine as accurately as possible the magnitude of the resultant force and the angle its direction makes with that of the force of 19 lbs. (13)

3. Draw a small circle (C) and let it represent a wheel fixed immoveably, with its plane vertical; suppose a thread to pass over the edge of the wheel; if there is no friction, and one end (A) of the thread carries a weight W, while the other end (B) is fastened to a hook in the floor vertically under C, what is the magnitude of the pull on the hook? If the hook were driven into a wall, so that CB is now horizontal, explain whether this makes any change in the pull on the hook.

Find, in each case, the pressure on the wheel. (12)

4. A rod AB rests horizontally on two points one under each end; its length is divided into three equal parts at points C and D; if weights of 12 lbs. and 18 lbs. are hung from C and D, find the pressure on A and on B, (a) neglecting the weight of the rod, (b) taking account of the fact that the rod is of uniform density and weighs 6 lbs. (12)

5. Define the centre of gravity of a body.

State what is the position of the centre of gravity:—(a) of a square lamina, (b) of a cylinder, (c) of a triangular lamina, (d) of a cone. In each case the body is of uniform density. (10)

6. Define the axis of a couple.

I attempt to turn the cork in a vertical bottle by a couple whose axis is vertical. If the force of the couple be 10 lbs. and its moment equal to 7·5 numerically, a foot being taken as the unit of length, find the arm of the couple in inches. If the length of the arm be increased 10 per cent., find by how much per cent. the force must be diminished so as to leave the moment unchanged. (15)

7. A board in shape an isosceles triangle has the base, which is lower than the vertex, fixed horizontally, and is moveable about the base as about a smooth hinge; it is kept in equilibrium by a string connecting the vertex with a fixed point, which is in the same vertical plane as the perpendicular from the vertex to the base. Construct a triangle of forces for the equilibrium, and show how to determine the reacting force of the hinge in magnitude and direction. (14)

8. Three million gallons of water flow daily over a fall, the height of which is 18 ft.; assuming that $\frac{2}{13}$ths of the power is wasted, what is the horse-power of a wheel worked by the fall? (A gallon of water weighs 10 lbs.)

A man being asked what is meant by a Horse-power, said that it meant 33,000 foot-pounds of work. State in what way the answer was wrong. (16)

9. Write down the usual formulæ of uniformly accelerated motion, and from them deduce a formula which does not involve the initial velocity.

A ball is thrown vertically upwards and in 3·125 seconds strikes a horizontal board with a velocity of 5·5 feet per second; neglecting the resistance of the air, find the height of the board. (14)

10. Two bodies, whose masses are 9 lbs. and 11 lbs., are connected by a fine thread, which passes over a perfectly smooth fixed wheel; find the increase of the velocity of either body in each second of the motion. Find the height through which the lighter body rises in the first three seconds of the motion. (12)

11. A body of mass m acquires a velocity v, when acted on (in the direction of the motion) by a force F through a space s. What is the equation of the work and energy? (10)

12. State what is meant by the centrifugal force of a body moving in a circle.

Write down the formula for the centrifugal force, when the mass of the body is m, its velocity v, and the radius of the circle r, and state what units must be used if the formula is to give the force in poundals.

The radius of the circle being 10 ft., find the velocity, when the centrifugal force of the body equals its weight. (16)

UNIVERSITY OF LONDON.

MATRICULATION (JUNE, 1900).

Mechanics.

1. Explain how a screw is a particular case of an inclined plane. Apply the principle of work to determine the force that will be exerted by a screw of pitch, p, when a moment, m, is applied to turn it.

2. A sailor is running across the deck of a ship which is heeling over at an angle of 30°. If he run in a direction perpendicular to the length of the ship at the rate of 5 yards per second and the ship be moving forward through the water at the rate of 8 miles an hour, how fast is the sailor moving with reference to the body of the water?

3. Suppose a force to act upon a body which is free to move only along a given straight line. How would you ascertain (a) by experiment, and (b) by geometry, how much of that force is effective towards producing motion along that line? Illustrate your answer by reference to a case in which the angle between the direction of the force and that of the line is equal to $\pi/6$ radians.

4. A man is pulling a boat by means of a rope. The boat weighs 200 lbs., and is resisted by the water with a force of 10 lbs. The rope is 40 feet long, and each foot weighs 1 lb. Calculate the force the man must exert (a) if the boat be moving uniformly, (b) if it be accelerated at the rate of 2 feet per second per second. In each case what is the difference between the force with which the man pulls the rope and that with which the rope pulls the boat?

5. By what experiments would you prove that the work that can be done by a moving body is proportional to (a) its mass, (b) the square of its velocity.

6. Define density. You are given some water, some turpentine, a piece of glass tube, a spirit lamp, and a foot rule, and are required to find the density of the turpentine; how would you do this?

7. When a body is partially immersed in water, through what point does the resultant upward thrust of the water act? As the body is turned about in the water, does this point always stay the same? Apply this to show that a uniform straight rod floating vertically in water is in a position of equilibrium.

ANSWERS.

Exercises I. a.

1. 74 units inclined at 62° and 72° to given forces.

2. 14·7 lbs.' wt. inclined at 22° and 44° to given forces.

3. 20 units inclined at 60° and 30° to given forces.

Exercises I. b.

1. 36·8 tons' wt. at 55° and 71° with given forces.

2. 40 at 30° and 120° with given forces.

Exercises I. c.

1. 14·1 and 10 lbs.' wt.
2. 20·8 and 41·6 lbs.' wt.

3. 8·6 grams' wt.
4. 20·3 grams' wt.

Exercises II.

3. 208 lbs.' wt. nearly.
4. 104 and 60 lbs.' wt.

7. 306 and 216 lbs.' wt.
10. 6·9 lbs.' wt., at 30° to the vertical.

11. 30·35 and 30 lbs.' wt.
12. 173·2 lbs.; 100 lbs.' wt.

16. 28·86 and 16·03 lbs.' wt.
17. $1\frac{1}{2}$ lbs.' wt.

18. 157°, 90°, and 113°.

19. 22 units; 39°; in direction opposite to resultant.

20. 25 lbs.
21. 12 and 10.
22. (1) $\frac{P\sqrt{3}}{2}$; (2) 0.

23. 7 lbs.' wt.; $3\sqrt{2}$ lbs.' wt.
25. $\frac{1}{3}\sqrt{3}$ lbs.' wt.; $\frac{2}{3}\sqrt{3}$ lbs.' wt.

Exercises III. a.

1. 7 lbs.
2. 1 lb.
3. $1\frac{1}{3}$ inches from O on first side.

Exercises III. b.

1. 5 lbs.' wt. in sense of forces acting at C; $AC=6$ ft., $CB=4$ ft.

2. 17 units in sense of forces at C; $AC=2\frac{16}{17}$, $CB=2\frac{1}{17}$.

3. 3 units in sense of the 10 at C; $AC=16\frac{2}{3}$, $CB=11\frac{2}{3}$.
4. 8 ozs.' wt. in sense of the 25 at C; $AC=8\frac{1}{2}$ ins., $CB=12\frac{1}{2}$ ins.
5. 25 tons' wt. in sense of 75; $AC=12$ ins., $BC=16$ ins.
6. $2P$ in sense of $3P$; $AC=6a$, $BC=10a$.
7. 41 units in sense opposite to given forces; $AC=3\frac{25}{41}$ ins., $BC=\frac{16}{41}$ ins.
8. 19 ozs.' wt. opposite to given forces; $AC=2$ ft. 9 ins., $BC=2$ ft.
9. A force of 3 units in direction of 7, on other side of 10, at distance 2 ft. 4 ins.
10. $43\frac{5}{7}$ ins.
11. $4a$.
12. $X=\dfrac{Py}{x+y}$; $Y=\dfrac{Px}{x+y}$.
13. $X=\dfrac{Py}{x-y}$; $Y=\dfrac{Px}{x-y}$.
14. $4\frac{1}{2}$ ins. and $7\frac{1}{2}$ ins. from P and Q respectively.
15. 1 ft. 4 ins. and 2 ft. 4 ins. from P and Q respectively.

Exercises IV.

1. $15\frac{3}{11}$ ins. from the 4 lb. wt.; $8\frac{8}{11}$ ins. from the 7 lb. wt.
2. 3840 lbs.' wt.
3. $8\frac{1}{3}$ lbs.' wt.
5. 12 ft. 6 ins. from A; 7·8 ozs.' wt.
6 Back wheel, $100\frac{1}{10}$ lbs.' wt.; front wheel, $53\frac{9}{10}$ lbs.' wt.
7. $20\frac{4}{11}$ lbs.' wt.
8. $4\frac{4}{7}$ ins.
9. 50 lbs.' wt.; 35 lbs.' wt.
10. ·1 lb. wt. about.
11. 48 units.
14. The string at B; $9\frac{1}{12}$ lbs.' wt.
16. Perpendicular distances between axis and directions of forces are in ratio of 4 to 5.
17. 225 lbs.' wt.

Exercises V. a.

1. $2\frac{11}{29}$ ins. from the 4 oz.
2. $\frac{1}{5}$ lb.
3. 11 inches from the end mentioned.
4. $3\frac{3}{7}$ ft.
5. 28 lbs.
6. 2·16 metres.
7. $186\frac{6}{29}$ cm.
8. Length = 13 ft.; C.G. 5 feet from A.
9. 8, 6·4, 6·4 ins.
12. 3, $\sqrt{10}$, $\sqrt{10}$.
13. At the junction of the rods.
15. In the line of centres, 2 ins. from the centre of the largest disc.
16. $9\frac{59}{169}$ units.

Exercises V. b.

1. $\frac{2}{55}$ cub. ft.
2. 6 kilograms.

Exercises VI. a.

1. $\frac{7}{10}AB$ and $\frac{1}{2}AB$. 2. In angle COD; each distance is $\frac{a}{6}$.

3. Let O be the centre and A the angular point omitted; C.G. is on AO produced at distance from $O=\frac{1}{5}AO$.

4. At G in BC, so that $BG=1\frac{12}{31}$.

6. $1\frac{7}{57}$ ins. from centre of middle sphere towards centre of that of radius 6 ins.

8. In the median AD of the triangle ABC at distance from A equal to $\frac{2(n^2+n+1)}{3n(n+1)}AD$.

Exercises VI. b.

1. 76 and $6\frac{2}{3}$ lbs.' wt. 3. 2·65 ins. 6. 44 and 38 ozs.' wt.

7. 3 ft. from cord whose tension is double that of the other.

8. $2\frac{1}{6}$ ft.

9. Let a be side of square; then C.G. is at a point in the straight line drawn through the middle point of the hypotenuse and at right angles to it, at a distance $\frac{11a}{30}$ from hypotenuse.

10. At a point G in the straight line joining the middle points A, B, of the arms, so that $AG=\frac{2}{3}GB$.

11. 10·4 ft. nearly.

12. $4\frac{2}{3}$ cms. and $2\frac{1}{2}$ cms. from sides (1, 10), (1, 2) respectively.

13. $1\frac{4}{5}$ ft. from end on which weight is placed.

14. Divide AB at D so that $AD=2DB$; then C.G. is at middle point of CD.

15. $3\sqrt{2}$ lbs.' wt.

16. At distances 8 ins. and 6 ins. from sides (1, 1), (1, 2) respectively.

17. In line joining centres of disc and hole, $\frac{7}{143}$ ins. from centre of disc.

18. $6\sqrt{3}$ stones' wt. and 30 stones' wt.

19. $\frac{5}{12}$ths of diagonal through 5. 20. $\frac{600}{119}\sqrt{119}$ lbs.' wt.

21. $\frac{1}{3}$rd of distance between middle points of AB, BC.

Exercises VII.

1. 1842 lbs.' wt. 2. 90,000 lbs.' wt. per sq. in.

3. 268,800 lbs.' wt. per sq. in.

Exercises VIII. a.

1. 7,200 ft.-lbs.; 2 mins. $10\frac{10}{11}$ secs. 2. 4,200 ft.-lbs.; $\frac{7}{55}$.

3. 158,400 ft.-lbs. 4. 22,500 ft.-lbs. 6. 35 ft.-lbs. per sec.

7. 400,000 ft.-lbs. 8. 18·32. 10. 22 mins. 44 secs.
11. $454\frac{6}{11}$. 12. 224 : 275. 13. $\frac{7}{2250}$.
14. $22{,}628\frac{4}{7}$ ft.-lbs. 15. $58\frac{19}{32}$. 18. 200 H.P.
19. 883·93. 20. 76·04. 24. $4\frac{1}{2}$ ft.-lbs.

Exercises VIII. b.

1. 336 ft.-lbs. 2. $75{,}428\frac{4}{7}$ ft.-lbs. 3. 9 ft.-lbs.
4. 5320 ft.-lbs. 5. $192\frac{6}{7}$. 6. $4{,}827\frac{3}{7}$.
8. 36·29. 9. 2424·8 ft.-lbs. 10. ·0052 ft.-lbs.
11. 2140 ft.-lbs. per min. 12. 20,160 ft.-lbs.; $\frac{1}{8}$th.
13. 56. 14. 1400 ft.-lbs.

Exercises IX. a.

1. $13\frac{1}{23}$ lbs.' wt. 2. 200 lbs.' wt. 3. $17\frac{12}{39}$ lbs.' wt.
4. $\frac{100}{103}$; $\frac{3}{103}$. 5. $3\frac{1}{3}$ ozs.' wt. 6. 125 lbs.' wt.
7. $\frac{9}{16}$ BC from B; pressure = 13·89 units, and acts at an angle of 86° to BC.

Exercises IX. b.

1. 60 and 80 lbs.' wt. 2. 75 and 125 lbs.' wt. 3. 12,000 ft.-lbs.
4. 3·2 yds. 6. 116·1 ft.-lbs. 7. 11·31 ft.
8. 6336. 12. (1) $\frac{2}{25}$; (2) $\frac{3}{25}$. 13. 0°.
14. 7 : 24. 15. ·34 cm. 16. 30 lbs.' wt.
17. $\frac{2fd - Wl}{2f - W}$. 18. 106 lbs.' wt.
19. $\sqrt{3}$ lbs.' wt. along the plane. 20. $\frac{112}{165}$.
21. 70·7 lbs.' wt. along the plane.

Exercises X. a.

1. 7 lbs.' wt. 2. 320 lbs. 3. 1 : 64. 4. 5.
5. 4; 160 lbs. 6. $37\frac{1}{3}$ lbs.' wt. 7. 20 lbs.' wt.
8. $17\frac{1}{3}$ lbs.' wt.; $69\frac{1}{3}$ lbs.' wt. 9. 28 grams. 11. $6\frac{1}{4}$ lbs.' wt.
12. $107\frac{1}{2}$ lbs.' wt. 15. $46\frac{2}{3}$ lbs.' wt. 16. 390 lbs. 17. 1200 lbs.
19. $5\frac{5}{12}$ lbs.' wt. 20. 4. 21. 4 lbs. 22. 3 lbs.
24. $\frac{2}{3}$. 25. $\frac{13}{34}$. 26. $\frac{21}{40}$. 27. 10.

Exercises X. b.

1. 25 lbs.' wt. 2. 10 ins. 3. 2 ft. 4. $452\frac{4}{7}$ ft.-lbs.
5. $20\frac{20}{21}$ ft.-lbs. 6. $\frac{112}{125}$. 7. 2·31 metres. 8. ·95 ins.
9. 1000 lbs.

Exercises XI. a.

1. 17 : 16 ; $3\frac{13}{17}$. **4.** 21 : 20. **5.** $2\frac{1}{90}$ lbs.
7. 2s. 8d. **8.** 8 : 9.

Exercises XI. b.

1. 22 ins. and 50 ins. from the fulcrum. **2.** 83 cms.; 2 cms.
4. 2 lbs. **5.** 6 lbs. and 3 lbs. **6.** $3\frac{3}{7}$, 6, and 8 ins.
9. $14\frac{26}{41}$ lbs.

Exercises XII.

2. $17\frac{1}{22}$ secs. **3.** 112 ft. per sec.; 128 ft. per sec.
4. $\frac{44}{135}$ ft. per sec. per sec.

Exercises XIII. a.

1. $3\frac{1}{3}$ ft. per sec. per sec. **2.** $-\frac{1}{2}$ cm. per sec. per sec.
3. $\frac{1}{200}$ cm. per sec. per sec. **4.** $-\frac{1}{60}$ ft. per sec. per sec.
5. $\frac{1}{15}$ ft. per sec. per sec. **6.** -5000 ft. per sec. per sec.
7. $-\frac{11}{15}$ ft. per sec. per sec. **8.** 560 ft. per sec. per sec.
9. 4 secs. **11.** 18 units.
12. 72 ft. per sec. per sec. **13.** 8·49 ft. per sec.
14. 50 ft. per sec. **15.** -3 ft. per sec. per sec.; 4 secs.
16. 5 ft. per sec. **18.** 1,712,500 ft. per sec. per sec.; $\frac{1}{6850}$ secs.
19. 200. **20.** $-\frac{7}{8}$ ft. per sec. per sec.
21. $12\frac{7}{9}$. **22.** 7 secs.
23. 1 ft. per sec. per sec. **24.** $-2,940,000$ ft. per sec. per sec.

Exercises XIII. b.

1. 324 ft. **3.** 36 ft. **4.** 3 secs.
5. 20 ft. per sec. **6.** 64 ft. **7.** $24\frac{1}{2}$ secs.
8. 1·92 secs. **9.** 88 ft. per sec. **10.** $36\frac{1}{2}$ secs.; 22,500 ft.
11. In $1\frac{1}{4}$ secs.; 25 ft. below the top. **12.** 57.
13. 1024 cms. **14.** In 1 sec.; 144 ft. **15.** 172 ft.
16. 5 units. **17.** 22 ft. per sec. **18.** $122\frac{1}{2}$ cms.
19. 58 ft.; 58 ft. per sec. **20.** 225 ft. below *B*. **21.** $28\frac{4}{5}$ ft. per sec.
22. 7·84 secs. **23.** $\frac{5}{108}$ miles per min. per min.
24. $\frac{1}{6}$ ft. per sec. per sec.; $\frac{1}{3}$ ft. per sec. per sec.
25. $22\frac{1}{2}$ secs.; $2\frac{82}{135}$ ft. per sec. per sec.

Exercises XIV. a.

1. $\frac{1}{25}$ ft.-sec. unit. 2. $64\frac{4}{15}$ poundals. 3. ·0328 grams.
4. 30·48. 5. 4536. 6. 26·4552.
7. ·0000723. 8. 2·54 grams.

Exercises XIV. b.

1. 37 mins. 20 secs. 2. 2800 dynes. 3. $51\frac{1}{5}$ cms.
4. 2 ft. per sec. 5. $2\frac{1}{2}$ ft. per sec. 6. 12.
7. 3 hrs. 28 mins. 20 secs. 8. 2 secs.

Exercises XV. a.

1. $10\frac{2}{3}$ units. 2. 4·905 kilos. 3. $\frac{1}{1120}$ ft.-sec. unit.
4. 125 lbs.' wt. 6. $\frac{7}{48}$.

Exercises XV. b.

1. $\frac{8}{9}$ cm. per sec. per sec.; $1\frac{7}{9}$ dynes.
3. $10\frac{2}{3}$ ft.; $\frac{2}{3}$ and $\frac{1}{3}$ lb. wt. 4. $\frac{5}{6}$ lb. wt.; $1\frac{1}{2}$ secs.
5. 25·9 secs. 6. $\frac{5}{6}$ lb. wt. 7. 2 lbs.
8. $\frac{100}{9599}$ lb. wt. 9. $6\frac{2}{5}$ ft. per sec. per sec.; $2\frac{2}{5}$ lbs.' wt.
10. 4 ft. 12. $2\frac{3}{4}$ secs. 13. 9·8 grams' wt.
14. $43\frac{1}{13}$ and $46\frac{2}{13}$ grams' wt. 15. $31\frac{1}{4}$ ft. per sec. per sec.
18. $\frac{g}{9}$ ft. per sec. per sec.; $4\frac{4}{9}$ lbs.' wt. 19. 3·95 secs.
20. $2\frac{10}{11}$ ft. per sec. per sec. down the plane; $1\frac{1}{11}$ lbs.' wt.
21. $\frac{16}{21}$ ft. per sec. per sec.; $5\frac{16}{21}$ lbs.' wt.
22. $\frac{13}{70}$ ft. per sec. per sec.; 112 lbs.' wt.

Exercises XV. c.

1. $1\frac{1}{31}$ ozs. 2. 93·1 grams. 3. 2 : 1.
4. 1 : 3. 5. 1 oz.; $15\frac{1}{2}$ ozs. 6. 9·05 secs.
7. (i.) 120 poundals; (ii.) 195 poundals. 9. $6133\frac{1}{3}$ lbs.' wt.
10. 2·8 lbs.' wt.; 17·07 ft. per sec. per sec. 11. 8 lbs.
12. 1·22 secs. 13. 8 ft. per sec. per sec.; $11\frac{1}{4}$ lbs.' wt.
14. $1\frac{1}{4}$ cms. per sec. per sec.
15. $-\frac{11}{1125}$ ft. per sec. per sec.; $6\frac{38}{45}$ lbs.' wt.
16. 2 ft. per sec. 17. 2 ft. per sec. 18. 1·78 secs.
19. 1600 units. 20. $3\frac{5}{9}$ ft. per sec. per sec.; $44\frac{4}{9}$ ft.

Exercises XVI. a.

1. 200 poundals. 2. 35 secs. 3. 7168 pounds.
4. $\frac{1}{57344}$ secs. 5. 120 poundals ; 6000 poundals.
6. At 3 ft. per sec. opposite to direction of force.
7. $2\frac{5}{6}$ lbs.' wt. 8. 20 ozs.' wt.

Exercises XVI. b.

1. 30 units. 2. 3,000,000 C.G.S. units. 3. 150 lbs.' wt.
4. $12\frac{1}{2}$ units. 5. $17\frac{1}{2}$ units.

Exercises XVI. c.

1. $4\frac{11}{16}$ ft. per sec. 2. 1154 ft. per sec.
3. The kilogram mass goes on moving in the same sense as before with velocity 9·75 metres per second ; 250 C.G.S. units of momentum.
4. $1\frac{1}{8}$ ft. per sec. in same sense as before. 6. $\frac{1}{320}$ sec.

Exercises XVII. a.

1. (1) 2800, $87\frac{1}{2}$; (2) 31250, $976\frac{9}{16}$; (3) 4480, 140.
2. (1) 40,000, 40·77 ; (2) 2,000,000,000,000, 2,039,000,000.
3. 44,037. 4. The first is 22·662 of the other.
6. $\frac{3}{4}$ inch. 7. $1\frac{1}{2}$ inches. 8. 1210.

Exercises XVII. b.

1. $7\frac{1}{2}$ lbs.' wt. 2. 4 ft. per sec. 3. 995 metres. 4. 495 metres.

Exercises XVII. c.

1. $410\frac{2}{3}$ lbs.' wt. 2. 25 yds. 3. 34·93 ft. per sec. (about).
4. 81·5 metres (about). 5. ·0498.

Exercises XVIII. a.

2. 11·3 ft. per sec. (about). 3. 17·5 ft.-lbs.; ·175 lbs.' wt.
4. $4\frac{3}{8}$ ft.-lbs. 5. 36·17 ft. tons. 6. $1\frac{1}{7}$ ft. per sec. 7. $5\frac{1}{4}$ ft.-lbs.
8. 30 ft. per sec. per sec. 9. 9·902 cms. per sec.

Exercises XVIII. b.

1. $222\frac{8}{11}$ foot-poundals 2. 2,400,000,000.
3. 3915 ft. per sec. 4. $6\frac{1}{2}$ ft. per sec.; $7\frac{59}{64}$ inches.
5. 16 ft. 7. $5\frac{1}{3}$ ft. per sec.; $1\frac{1}{3}$ ft.
10. 8 ft. per sec.; 5 ft. per sec.

11. 6050 ft.-poundals; 6050 ft.-poundals.

12. $787\frac{1}{2}$ ft.-poundals; 224 ft.-poundals; $1\frac{359}{768}$ lbs.' wt.

13. 180,000 ft.-poundals; 1875 ft. **14.** $16\sqrt{5}$ ft. per sec.

15. $2\frac{1}{4}$ ft.-lbs. **16.** 30 ft.-poundals.

17. 12,500 ft.-poundals; 500 F.P.S. units; 25 ft. per sec.

18. $1\frac{3}{5}$ ft.-poundals; $2\sqrt{\frac{2}{5}}$ ft. per sec.

Exercises XIX. a.

1. 25 ft. per sec.

2. In a direction making 120° with that in which the carriage is moving.

3. 30°. **6.** $V\sqrt{3}$ in the direction of the bisector of the angle.

Exercises XIX. b.

1. 120 ft. **2.** 226·3 ft. per sec.

Exercises XIX. c.

1. $\frac{11}{75600}$. **3.** 17·93. **4.** 7·41 lbs.' wt. **5.** $9\frac{3}{16}$ ins.

8. 26 cms. per sec. per sec. (nearly) at an angle of 74° to the easterly direction.

9. 16 ft. per sec **11.** 494 poundals nearly.

Exercises XX.

1. 9·785 ins. **4.** 3 mins. 36 secs. **5.** 111 lbs. 10·4 oz.

7. 56 ins. (about). **8.** 36·47 ins.

Examination Papers.

South Kensington, 1900.

1. 14·69 lbs. per sq. in. **2.** 31 lbs.; 28°. **3.** W; no; $2W$; $W\sqrt{2}$.

4. (*a*) 14 lbs.' wt.; 16 lbs.' wt.; (*b*) 17 lbs.' wt.; 19 lbs.' wt.

6. 9 ins.; $9\frac{1}{11}$ %. **8.** $9\frac{8}{13}$ H.P. **9.** $173\frac{7}{16}$ ft.

10. $3\frac{1}{5}$ ft. per sec. per sec.; $14\frac{2}{5}$ ft. **12.** $8\sqrt{5}$ ft. per sec.

London Matriculation, June, 1900.

2. 17·50 ft. per sec.

4. (*a*) 10 lbs.' wt.; no difference; (*b*) 25 lbs.' wt.; difference of $2\frac{1}{2}$ lbs.' wt.

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